

# Chapter 17

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## **Linear State Space Models**

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There are many alternative model formats that can be used for linear dynamic systems. In simple SISO problems, any representation is probably as good as any other. However, as we move to more complex problems (*especially multivariable problems*), it is desirable to use special model formats. One of the most flexible and useful structures is the state space model.

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We will examine linear state space models in a little more depth for the SISO case. Many of the ideas will carry over to the MIMO case which we will study later. In particular we will study

- ❖ similarity transformations and equivalent state representations,
- ❖ state space model properties:
  - ◆ controllability, reachability, and stabilizability,
  - ◆ observability, reconstructability, and detectability,
- ❖ special (*canonical*) model formats.

# Linear Continuous-Time State Space Models

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A continuous-time linear time-invariant state space model takes the form

$$\begin{aligned}\dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t) & x(t_0) &= x_0 \\ y(t) &= \mathbf{C}x(t) + \mathbf{D}u(t)\end{aligned}$$

where  $x \in \mathbb{R}^n$  is the state vector,  $u \in \mathbb{R}^m$  is the control signal,  $y \in \mathbb{R}^p$  is the output,  $x_0 \in \mathbb{R}^n$  is the state vector at time  $t = t_0$  and  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ , and  $\mathbf{D}$  are matrices of appropriate dimensions.

# Similarity Transformations

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It is readily seen that the definition of the state of a system is nonunique. Consider, for example, a linear transformation of  $x(t)$  to  $\bar{x}(t)$  defined as

$$\bar{x}(t) = \mathbf{T}^{-1}x(t) \qquad x(t) = \mathbf{T}\bar{x}(t)$$

where  $\mathbf{T}$  is any nonsingular matrix, called a similarity transformation.

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The following alternative state description is obtained

$$\begin{aligned}\dot{\bar{x}}(t) &= \bar{\mathbf{A}} \bar{x}(t) + \bar{\mathbf{B}} u(t) & \bar{x}(t_o) &= \mathbf{T}^{-1} x_o \\ y(t) &= \bar{\mathbf{C}} \bar{x}(t) + \bar{\mathbf{D}} u(t)\end{aligned}$$

where

$$\bar{\mathbf{A}} \triangleq \mathbf{T}^{-1} \mathbf{A} \mathbf{T} \quad \bar{\mathbf{B}} \triangleq \mathbf{T}^{-1} \mathbf{B} \quad \bar{\mathbf{C}} \triangleq \mathbf{C} \mathbf{T} \quad \bar{\mathbf{D}} \triangleq \mathbf{D}$$

The above model is an equally valid description of the system.

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An illustration, say that the matrix  $\mathbf{A}$  can be diagonalized by a similarity transformation  $\mathbf{T}$ ; then

$$\bar{\mathbf{A}} = \mathbf{\Lambda} \triangleq \mathbf{T}^{-1} \mathbf{A} \mathbf{T}$$

where if  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of  $\mathbf{A}$ , then

$$\mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

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Because  $\Lambda$  is diagonal, we have

$$\bar{x}_i(t) = e^{\lambda_i(t-t_o)}\bar{x}_o + \int_{t_o}^t e^{\lambda_i(t-\tau)}\bar{b}_i u(\tau) d\tau$$

where the subscript  $i$  denotes the  $i^{\text{th}}$  component of the state vector.



# Example

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$$\mathbf{A} = \begin{bmatrix} -4 & -1 & 1 \\ 0 & -3 & 1 \\ 1 & 1 & -3 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \quad \mathbf{C} = [-1 \quad -1 \quad 0] \quad \mathbf{D} = 0$$

The matrix  $\mathbf{T}$  can also be obtained by using the MATLAB command **eig**, which yields

$$\mathbf{T} = \begin{bmatrix} 0.8018 & 0.7071 & 0.0000 \\ 0.2673 & -0.7071 & 0.7071 \\ -0.5345 & -0.0000 & 0.7071 \end{bmatrix}$$

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We obtain the similar state space description given by

$$\begin{aligned}\bar{\mathbf{A}} = \mathbf{\Lambda} &= \begin{bmatrix} -5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}; & \bar{\mathbf{B}} &= \begin{bmatrix} 0.0 \\ -1.414 \\ 0.0 \end{bmatrix}; \\ \bar{\mathbf{C}} &= [-0.5345 \quad -1.4142 \quad 0.7071] & \bar{\mathbf{D}} &= 0\end{aligned}$$

# Transfer Functions Revisited

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The solution to the state equation model can be obtained via

$$\begin{aligned} Y(s) &= [\bar{\mathbf{C}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{B}} + \bar{\mathbf{D}}]U(s) + \bar{\mathbf{C}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{x}(0) \\ &= [\mathbf{CT}(s\mathbf{I} - \mathbf{T}^{-1}\mathbf{AT})^{-1}\mathbf{T}^{-1}\mathbf{B} + \mathbf{D}]U(s) + \mathbf{CT}(s\mathbf{I} - \mathbf{T}^{-1}\mathbf{AT})^{-1}\mathbf{T}^{-1}x(0) \\ &= [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]U(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}x(0) \end{aligned}$$

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We thus see that different choices of state variables lead to different internal descriptions of the model, but to the same input-output model, because the system transfer function can be expressed in either of the two equivalent fashions.

$$\bar{\mathbf{C}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{B}} + \bar{\mathbf{D}} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

for any nonsingular  $\mathbf{T}$ .

# From Transfer Function to State Space Representation

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We have seen above how to go from a state space description to the corresponding transfer function. The converse operation leads to the following question:

Given a transfer function  $G(s)$ , how can a state representation for this system be obtained?

# Development

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Consider a transfer function  $G(s) = B(s)/A(s)$ . We can then write

$$Y(s) = \sum_{i=1}^n b_{i-1} V_i(s) \quad \text{where} \quad V_i(s) = \frac{s^{i-1}}{A(s)} U(s)$$

We note from the above definitions that

$$v_i(t) = \mathcal{L}^{-1} [V(s)] = \frac{dv_{i-1}(t)}{dt} \quad \text{for} \quad i = 1, 2, \dots, n$$

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We can then choose, as state variables,  $x_i(t) = v_i(t)$ , which lead to the following state space model for the system.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$
$$\mathbf{C} = [b_0 \quad b_1 \quad b_2 \quad \cdots \quad b_{n-1}] \quad \mathbf{D} = 0$$

The above model has a special form. We will see later that any *completely controllable* system can be expressed in this way. Before we do this, we need to introduce the idea of controllability.

# Controllability and Stabilizability

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An important question that lies at the heart of control using state space models is whether we can steer the state via the control input to certain locations in the state space. Technically, this property is called controllability or reachability. A closely related issue is that of stabilizability. We will begin with controllability.



# Controllability

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The issue of controllability concerns whether a given initial state  $x_0$  can be steered to the origin in finite time using the input  $u(t)$ .

Formally, we have the following:

**Definition 17.1:** A state  $x_0$  is said to be controllable if there exists a finite interval  $[0, T]$  and an input  $\{u(t), t \in [0, T]\}$  such that  $x(T) = 0$ . If all states are controllable, then the system is said to be completely controllable.

# Reachability

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A related concept is that of reachability. This concept is sometimes used in discrete-time systems. It is formally defined as follows:

**Definition 17.2:** A state  $\bar{x} \neq 0$  is said to be reachable (*from the origin*) if, given  $x(0) = 0$ , there exist a finite time interval  $[0, T]$  and an input  $\{u(t), t \in [0, T]\}$  such that  $x(T) = \bar{x}$ . If all states are reachable, the system is said to be completely reachable.

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For continuous, time-invariant, linear systems, there is no distinction between complete controllability and complete reachability. However, the following example illustrates that there is a subtle difference in discrete time.

Consider the following shift-operator state space model:

$$x[k + 1] = 0$$

This system is obviously completely controllable: the state immediately goes to the origin. However, no nonzero state is reachable.

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In view of the subtle distinction between controllability and reachability in discrete time, we will use the term *controllability* in the sequel to cover the stronger of the two concepts. The discrete-time proofs for the results presented below are a little easier. We will thus prove the results on the following discrete-time (*delta-domain*) model:

$$\begin{aligned}\delta x[k] &= \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k] \\ y[k] &= \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]\end{aligned}$$

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Our next step will be to derive a simple algebraic test for controllability that can easily be applied to a given state space model. In deriving this result, we will use a result from linear algebra known as the Cayley-Hamilton Theorem.

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**Theorem 17.1:** (*Cayley-Hamilton theorem*). Every matrix satisfies its own characteristic equation - i.e., if

$$\det(s\mathbf{I} - \mathbf{A}) = s^n + a_{n-1}s^{n-1} + \dots + a_0$$

then

$$\mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \dots + a_0\mathbf{I} = 0$$

*Proof:* See the book.

# Test for Controllability

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**Theorem 17.2:** Consider the state space model

$$\delta x[k] = \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k]$$

$$y[k] = \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]$$

(i) The set of all controllable states is the range space of the controllability matrix  $\Gamma_c[\mathbf{A}, \mathbf{B}]$ , where

$$\Gamma_c[\mathbf{A}, \mathbf{B}] \triangleq [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$$

(ii) The model is completely controllable if and only if where  $\Gamma_c[\mathbf{A}, \mathbf{B}]$  has full row rank.

*Proof:* Uses Cayley-Hamilton Theorem - see book.

# Example 17.5

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Consider the state space model

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The controllability matrix is given by

$$\Gamma_c[\mathbf{A}, \mathbf{B}] = [\mathbf{B}, \mathbf{AB}] = \begin{bmatrix} 1 & -4 \\ -1 & -2 \end{bmatrix}$$

Clearly,  $\text{rank } \Gamma_c[\mathbf{A}, \mathbf{B}] = 2$ ; thus, the system is completely controllable.



# Example 17.6

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For

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The controllability matrix is given by:

$$\Gamma_c[\mathbf{A}, \mathbf{B}] = [\mathbf{B}, \mathbf{AB}] = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

$\text{Rank } \Gamma_c[\mathbf{A}, \mathbf{B}] = 1 < 2$ ; thus, the system is not completely controllable.

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Although we have derived the above result by using the delta model, it holds equally for shift and/or continuous-time models.

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We see that controllability is a black and white issue: a model either is completely controllable or it is not. Clearly, to know that something is uncontrollable is a valuable piece of information. However, to know that something is controllable really tells us nothing about the *degree* of controllability, i.e., about the difficulty that might be involved in achieving a certain objective. The latter issue lies at the heart of the fundamental design trade-offs in control that were the subject of Chapters 8 and 9.

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If a system is not completely controllable, it can be decomposed into a controllable and a completely uncontrollable subsystem, as explained below.

# Controllable Decompositon

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**Lemma 17.1:** Consider a system having  $\text{rank}\{\Gamma_c[\mathbf{A}, \mathbf{B}]\} = k < n$ ; then there exists a similarity transformation  $T$  such that  $\bar{x} = \mathbf{T}^{-1}x$ ,

$$\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}; \quad \bar{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}$$

and  $\bar{\mathbf{A}}, \bar{\mathbf{B}}$  have the form

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ 0 & \bar{\mathbf{A}}_{nc} \end{bmatrix}; \quad \bar{\mathbf{B}} = \begin{bmatrix} \bar{\mathbf{B}}_c \\ 0 \end{bmatrix}$$

where  $\bar{\mathbf{A}}_c$  has dimension  $k$  and  $(\bar{\mathbf{A}}_c, \bar{\mathbf{B}}_c)$  is completely controllable.

*Proof:* See the book.

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The above result has important consequences regarding control. To appreciate this, express the (*transformed*) state and output equations in partitioned form as

$$\delta \begin{bmatrix} \bar{x}_c[k] \\ \bar{x}_{nc}[k] \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{nc} \end{bmatrix} \begin{bmatrix} \bar{x}_c[k] \\ \bar{x}_{nc}[k] \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} \bar{\mathbf{C}}_c & \bar{\mathbf{C}}_{nc} \end{bmatrix} \begin{bmatrix} \bar{x}_c[k] \\ \bar{x}_{nc}[k] \end{bmatrix} + \mathbf{D}u[k]$$

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A pictorial representation of these equations is shown in Figure 17.1.

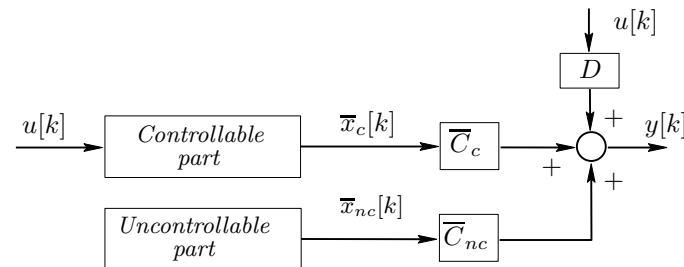


Figure 17.1: *Controllable-uncontrollable decomposition*

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We see that caution must be exercised when controlling a system (*or designing a controller with a model that is not completely controllable*), because the output has a component  $\mathbf{C}_{nc} \bar{\mathbf{x}}_{nc} [k]$  that does not depend on the manipulated input  $u[k]$ .



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The controllable subspace of a state space model is composed of all states generated through every possible linear combination of the states in  $\bar{x}_c$ . The stability of this subspace is determined by the location of the eigenvalues of  $\mathbf{A}_{nc}$ .

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The uncontrollable subspace of a state space model is composed of all states generated through every possible linear combination of the states in  $\bar{x}_{nc}$ . The stability of this subspace is determined by the location of the eigenvalues of  $\mathbf{A}_{nc}$ .

# Stabilizability

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A state space model is said to be *stabilizable* if its uncontrollable subspace is stable.

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A fact that we will find useful in what follows is that, if the system is completely controllable, there exist similarity transformations that convert it into special forms, known as *canonical forms*. This is established in the following two lemmas.

# Controllability Canonical Form

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**Lemma 17.2:** Consider a completely controllable state space model for a SISO system. Then there exists a similarity transformation that converts the state space model into the following controllability-canonical form:

$$\mathbf{A}' = \begin{bmatrix} 0 & 0 & \dots & 0 & -\alpha_0 \\ 1 & 0 & \dots & 0 & -\alpha_1 \\ 0 & 1 & \dots & 0 & -\alpha_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\alpha_{n-1} \end{bmatrix} \quad \mathbf{B}' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where  $\lambda^n + \alpha_{n-1}\lambda^{n-1} + \dots + \alpha_1\lambda + \alpha_0 = \det(\lambda\mathbf{I} - \mathbf{A})$  is the characteristic polynomial of  $\mathbf{A}$ .

*Proof:* See the book.

# Controller - Canonical Form

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**Lemma 17.3:** Consider a completely controllable state space model for a SISO system. Then there exists a similarity transformation that converts the state space model into the following controller-canonical form:

$$\mathbf{A}'' = \begin{bmatrix} -\alpha_{n-1} & -\alpha_{n-2} & \dots & -\alpha_1 & -\alpha_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad \mathbf{B}'' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where  $\lambda^n + \alpha_{n-1}\lambda^{n-1} + \dots + \alpha_1\lambda + \alpha_0 = \det(\lambda\mathbf{I} - \mathbf{A})$  is the characteristic polynomial of  $\mathbf{A}$ .

*Proof:* See the book.

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Finally, we remark that, as we have seen in Chapter 10, it is very common indeed to employ uncontrollable models in control-system design. This is because they are a convenient way of describing various commonly occurring disturbances. For example, a constant disturbance can be modeled by the following state space model:

$$\dot{x}_d = 0$$

which is readily seen to be uncontrollable and, indeed, nonstabilizable.

# Observability and Detectability

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Consider again the state space model

$$\begin{aligned}\delta x[k] &= \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k] \\ y[k] &= \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]\end{aligned}$$

In general, the dimension of the observed output,  $y$ , can be less than the dimension of the state,  $x$ . However, one might conjecture that, if one observed the output over some nonvanishing time interval, then this might tell us something about the state. The associated properties are called observability (*or reconstructability*). A related issue is that of detectability. We begin with observability.



# Observability

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Observability is concerned with the issue of what can be said about the state when one is given measurements of the plant output.

A formal definition is as follows:

**Definition 17.6:** The state  $x_0 \neq 0$  is said to be unobservable if, given  $x(0) = x_0$ , and  $u[k] = 0$  for  $k \geq 0$ , then  $y[k] = 0$  for  $k \geq 0$ . The system is said to be completely observable if there exists no nonzero initial state that it is unobservable.

# Reconstructability

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A concept related to observability is that of reconstructability. This concept is sometimes used in discrete-time systems. Reconstructability is concerned with what can be said about  $x(T)$ , on the basis of the past values of the output, i.e.,  $y[k]$  for  $0 \leq k \leq T$ . For linear time-invariant continuous-time systems, the distinction between observability and reconstructability is unnecessary. However, the following example illustrates that, in discrete time, the two concepts are different.

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Consider

$$\begin{aligned}x[k + 1] &= 0 & x[0] &= x_o \\ y[k] &= 0\end{aligned}$$

this system is clearly reconstructable for all  $T \geq 1$ , because we know for certain that  $x[T] = 0$  for  $T \geq 1$ . However, it is completely unobservable, because  $y[k] = 0 \forall k$ , irrespective of the value of  $x_0$ .

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In view of the subtle difference between observability and reconstructability, we will use the term observability in the sequel to cover the stronger of the two concepts.

# Test for Observability

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A test for observability of a system is established in the following theorem.

**Theorem 17.3:** Consider the state model

$$\delta x[k] = \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k]$$

$$y[k] = \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]$$

- (i) The set of all unobservable states is equal to the null space of the observability matrix  $\Gamma_o[\mathbf{A}, \mathbf{C}]$ , where

$$\Gamma_o[\mathbf{A}, \mathbf{C}] \triangleq \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}$$

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(ii) The system is completely observable if and only if  $\Gamma_0[\mathbf{A}, \mathbf{C}]$ , has full column rank  $n$ .

*Proof:* See the book.

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As for controllability, the above result also applies to continuous-time and discrete (*shift*) operator models.

# Example 17.1

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Consider the following state space model:

$$\mathbf{A} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \mathbf{C} = [1 \quad -1]$$

Then

$$\Gamma_o[\mathbf{A}, \mathbf{C}] = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -4 & -2 \end{bmatrix}$$

Hence,  $\text{rank } \Gamma_o[\mathbf{A}, \mathbf{C}] = 2$ , and the system is completely observable.



# Example 17.8

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Consider

$$\mathbf{A} = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \mathbf{C} = [1 \quad -1]$$

Here

$$\Gamma_o[\mathbf{A}, \mathbf{C}] = \begin{bmatrix} 1 & -1 \\ -2 & -2 \end{bmatrix}$$

Hence,  $\text{rank } \Gamma_o[\mathbf{A}, \mathbf{C}] = 1 < 2$ , and the system is not completely observable.

# Duality

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We see a remarkable similarity between the results in Theorem 17.2 and in Theorem 17.3. We can formalize this as follows:

**Theorem 17.4** (*Duality*). Consider a state space model described by the 4-tuple  $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ . Then the system is completely controllable if and only if the dual system  $(\mathbf{A}^T, \mathbf{C}^T, \mathbf{B}^T, \mathbf{D}^T)$  is completely observable.

# Observable Decomposition

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The above theorem can often be used to go from a result on controllability to one on observability, and vice versa. For example, the dual of Lemma 17.1 is the following:

**Lemma 17.4:** If  $\text{rank}\{\Gamma_0[\mathbf{A}, \mathbf{C}]\} = k < n$ , there exists a similarity transformation  $T$  such that with  $\bar{x} = \mathbf{T}^{-1}x$ ,  $\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ ,  $\bar{\mathbf{C}} = \mathbf{C}\mathbf{T}$ , then  $\bar{\mathbf{C}}$  and  $\bar{\mathbf{A}}$  take the form

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}_o & 0 \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{no} \end{bmatrix} \quad \bar{\mathbf{C}} = [\bar{\mathbf{C}}_o \quad 0]$$

where  $\bar{\mathbf{A}}_o$  has dimension  $k$  and the pair  $(\bar{\mathbf{C}}_o \bar{\mathbf{A}}_o)$  is completely observable.

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The above result has a relevance similar to that of the controllability property and the associated decomposition. To appreciate this, we apply the dual of Lemma 17.1 to express the (*transformed*) state and output equations in partitioned form as

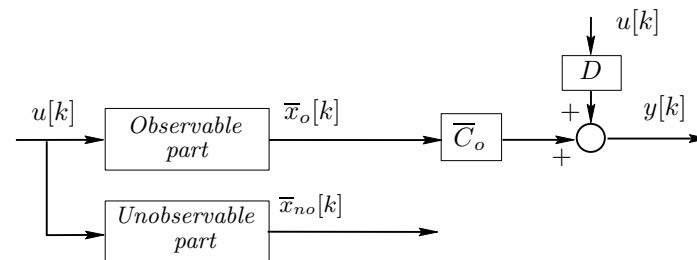
$$\delta \begin{bmatrix} \bar{x}_o[k] \\ \bar{x}_{no}[k] \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_o & 0 \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{n0} \end{bmatrix} \begin{bmatrix} \bar{x}_o[k] \\ \bar{x}_{no}[k] \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_o \\ \bar{\mathbf{B}}_{no} \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} \bar{\mathbf{C}}_o & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_o[k] \\ \bar{x}_{no}[k] \end{bmatrix} + \mathbf{D}u[k]$$

A pictorial description of these equations is shown on the next slide.

## Figure 17.2: *Observable-unobservable decomposition*

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The observable subspace of a plant is composed of all states generated through every possible linear combination of the states in  $\bar{x}_0$ . The stability of this subspace is determined by the location of the eigenvalues of  $\bar{\mathbf{A}}_0$ .

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The unobservable subspace of a plant is composed of all states generated through every possible linear combination of the states in  $\bar{x}_{n0}$ . The stability of this subspace is determined by the location of the eigenvalues of  $\bar{\mathbf{A}}_{n0}$ .

# Detectability

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A plant is said to be *detectable* if its unobservable subspace is stable.



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We remarked earlier that noncontrollable (*indeed nonstabilizable*) models are frequently used in control-system design. This is not true for nondetectable models. Essentially all models used in the sequel can be taken to be detectable, without loss of generality.

# Observer Canonical Form

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There are also duals of the canonical forms given in Lemmas 17.2 and 17.3. For example the dual of Lemma 17.3 is

**Lemma 17.5:** Consider a completely observable SISO system given by

$$\begin{aligned}\delta x[k] &= \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k] \\ y[k] &= \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]\end{aligned}$$

Then there exists a similarity transformation that converts the model to the observer-canonical form

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$$\delta x'[k] = \begin{bmatrix} -\alpha_{n-1} & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1 \\ -\alpha_0 & 0 & & 0 \end{bmatrix} x'[k] + \begin{bmatrix} b_{n-1} \\ \vdots \\ \vdots \\ b_0 \end{bmatrix} u[k]$$

$$y[k] = [1 \quad 0 \quad \dots \quad 0] x'[k] + \mathbf{D}u[k]$$

# Canonical Decomposition

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Further insight into the structure of linear dynamical systems is obtained by considering those systems that are only partially observable or controllable. These systems can be separated into completely observable and completely controllable systems.

The two results of Lemmas 17.1 and 17.4 can be combined as on the next slide.

# Canonical Decomposition Theorem

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**Theorem 17.5:** (*Canonical Decomposition Theorem*). Consider a system described in state space form. Then, there always exists a similarity transformation  $T$  such that the transformed model for  $\bar{x} = \mathbf{T}^{-1}x$  takes the form

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}_{co} & \mathbf{0} & \bar{\mathbf{A}}_{13} & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{22} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{33} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{34} & \bar{\mathbf{A}}_{44} \end{bmatrix}; \quad \bar{\mathbf{B}} = \begin{bmatrix} \bar{\mathbf{B}}_1 \\ \bar{\mathbf{B}}_2 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}; \quad \bar{\mathbf{C}} = [\bar{\mathbf{C}}_1 \quad \mathbf{0} \quad \bar{\mathbf{C}}_2 \quad \mathbf{0}]$$

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Where

- (i) The subsystem  $[ \bar{\mathbf{A}}_{c0}, \bar{\mathbf{B}}_1, \bar{\mathbf{C}}_1 ]$  is both completely controllable and completely observable and has the same transfer function as the original system.

---

(ii) The subsystem

$$\begin{bmatrix} \overline{\mathbf{A}}_{co} & \mathbf{0} \\ \overline{\mathbf{A}}_{21} & \overline{\mathbf{A}}_{22} \end{bmatrix}, \begin{bmatrix} \overline{\mathbf{B}}_1 \\ \overline{\mathbf{B}}_2 \end{bmatrix}, [\overline{\mathbf{C}}_1 \quad \mathbf{0}]$$

is completely controllable.

---

(iii) The subsystem

$$\begin{bmatrix} \overline{\mathbf{A}}_{co} & \overline{\mathbf{A}}_{13} \\ \mathbf{0} & \overline{\mathbf{A}}_{33} \end{bmatrix}, \begin{bmatrix} \overline{\mathbf{B}}_1 \\ 0 \end{bmatrix}, [\overline{\mathbf{C}}_1 \quad \overline{\mathbf{C}}_2]$$

is completely observable.



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The canonical decomposition described above leads to

**Lemma 17.6:** Consider the transfer-function matrix  $\mathbf{H}(s)$  satisfying

$$Y(s) = \mathbf{H}(s)U(s)$$

Then

$$\mathbf{H} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \overline{\mathbf{C}}_1(s\mathbf{I} - \overline{\mathbf{A}}_{co})^{-1}\overline{\mathbf{B}}_1 + \mathbf{D}$$

where  $\overline{\mathbf{C}}_1$ ,  $\overline{\mathbf{A}}_{co}$ , and  $\overline{\mathbf{B}}_1$  correspond to the observable and controllable part of the model.

---

Lemma 17.6 shows that the uncontrollable and the unobservable parts of a linear system do not appear in the transfer function. Conversely, given a transfer function, it is possible to generate a state space description that is both completely controllable and observable. We then say that this state description is a minimal realization of the transfer function. As mentioned earlier, nonminimal models are frequently used in control-system design to include disturbances.

---

Controllability depends on the structure of the input ports: where, in the system, the manipulable inputs are applied. Thus the states of a given subsystem might be uncontrollable for one given input but completely controllable for another. This distinction is of fundamental importance in control-system design, because not all plant inputs can be manipulated (*consider, for example, disturbances*) to steer the plant to reach certain states.

---

Similarly, observability depends on which outputs are being considered. Certain states may be unobservable from a given output, but they may be completely observable from some other output. This also has a significant impact on output-feedback control systems, because some states might not appear in the plant output being measured and fed back. However, they could appear in crucial internal variables and thus be important to the control problem.

# Pole-Zero Cancellation and System Properties

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The system properties described above are also intimately related to issues of pole-zero cancellations. To facilitate the subsequent development, we introduce the following test, which is useful for studying issues of controllability and observability.

# PBH Test

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**Lemma 17.7:** (*PBH Test*). Consider a state space model  $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ .

(i) The system is not completely observable if and only if there exist a nonzero vector  $x \in \mathbb{C}^n$  and a scalar  $\lambda \in \mathbb{C}$  such that

$$\mathbf{A}x = \lambda x$$

$$\mathbf{C}x = 0$$

(ii) The system is not completely controllable if and only if there exist a nonzero vector  $x \in \mathbb{C}^n$  and a scalar  $\lambda \in \mathbb{C}$  such that

$$x^T \mathbf{A} = \lambda x^T$$

$$x^T \mathbf{B} = 0$$

*Proof:* See the book.

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We will use the preceding result to study the system properties of cascaded systems.

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Consider the cascaded system shown below.

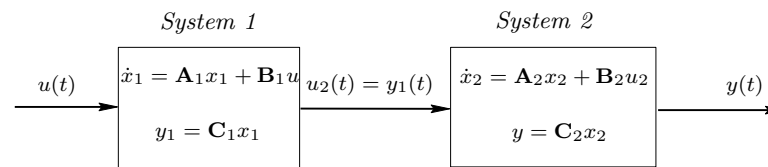


Figure 17.3: *Pole-zero cancellation*



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We assume that  $u(t)$ ,  $u_2(t)$ ,  $y_1(t)$ ,  $y(t) \in \mathbb{R}$ , that both subsystems are minimal, and that

System 1 has a zero at  $\alpha$  and pole at  $\beta$ ,

System 2 has a pole at  $\alpha$  and zero at  $\beta$ .

Then the combined model has the property that

- (a) The system pole at  $\beta$  is not observable from  $Y$ , and
- (b) The system pole at  $\alpha$  is not controllable from  $u$ .

The above results are readily established using the PBH test - see the book.

# Summary

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- ❖ State variables are system internal variables, upon which a full model for the system behavior can be built. The state variables can be ordered in a state vector.
- ❖ Given a linear system, the choice of state variables is not unique - however,
  - ◆ the minimal dimension of the state vector is a system invariant,
  - ◆ there exists a nonsingular matrix that defines a similarity transformation between any two state vectors, and
  - ◆ any designed system output can be expressed as a linear combination of the state variables and the inputs.

- 
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- ❖ For linear, time-invariant systems, the state space model is expressed in the following equations:

$$\begin{aligned} \textit{continuous-time systems} \quad \dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}x(t) + \mathbf{D}u(t) \end{aligned}$$

$$\begin{aligned} \textit{discrete-time systems, shift form} \quad x[k+1] &= \mathbf{A}_q x[k] + \mathbf{B}_q u[k] \\ y[k] &= \mathbf{C}_q x[k] + \mathbf{D}_q u[k] \end{aligned}$$

$$\begin{aligned} \textit{discrete-time systems, delta form} \quad \delta x[k] &= \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k] \\ y[k] &= \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k] \end{aligned}$$

- 
- ❖ Stability and natural response characteristics of the system can be studied from the eigenvalues of the matrix  $\mathbf{A}$  or  $(\mathbf{A}_q, \mathbf{A}_\delta)$ .
  - ❖ State space models facilitate the study of certain system properties that are paramount in the solution to the control-design problem. These properties relate to the following questions:
    - ◆ By proper choice of the input  $u$ , can we steer the system state to a desired state (*point value*)? (*controllability*)
    - ◆ If some states are uncontrollable, will these states generate a time-decaying component? (*stabilizability*)
    - ◆ If one knows the input,  $u(t)$ , for  $t \geq t_0$ , can we infer the state at time  $t = t_0$  by measuring the system output,  $y(t)$ , for  $t \geq t_0$ ? (*observability*)
    - ◆ If some of the states are unobservable, do these states generate a time-decaying signal? (*detectability*)

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- ❖ Controllability tells us about the feasibility of attempting to control a plant.
  - ❖ Observability tells us about whether it is possible to know what is happening inside a given system by observing its outputs.
  - ❖ The above system properties are system invariants. However, changes in the number of inputs, in their injection points, in the number of measurements, and in the choice of variables to be measured can yield different properties.

- 
- ❖ A transfer function can always be derived from a state space model.
  - ❖ A state space model can be built from a transfer-function model. However, only the completely controllable and observable part of the system is described in that state space model. Thus *the transfer-function model might be only a partial description of the system.*

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- ❖ The properties of individual systems do not necessarily translate unmodified to composed systems. In particular, given two systems completely observable and controllable, their cascaded connection
    - ◆ is not completely observable if a pole of the first system coincides with a zero of the second system (pole-zero cancellation),
    - ◆ is not detectable if the pole-zero cancellation affects an unstable pole,
    - ◆ is not completely controllable if a zero of the first system coincides with a pole of the second system (zero-pole cancellation), and
    - ◆ is not stabilizable if the zero-pole cancellation affects a NMP zero.

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- ❖ This chapter provides a foundation for the design requirement that one should never attempt to cancel unstable poles and zeros.