Chapter 18 - Solved Problems

Solved Problem 18.1. *A continuous time system has transfer function, $G_o(s)$, given by*

$$G_o(s) = \frac{2}{(s-1)(s+2)(s+3)} = \frac{2}{s^3+5s^2+s-6} \tag{1}$$

Find a state space realization $(\mathbf{A_o}, \mathbf{B_o}, \mathbf{C_o}, \mathbf{D_o})$. Then find state feedback gain, $\mathbf{K}$, such that the closed loop poles are located at -4, -2 and -3.

Solutions to Solved Problem 18.1

Solved Problem 18.2. *Consider a system with input $u(t)$ and output $y(t)$ having transfer function given by*

$$G_o(s) = \frac{B_o(s)}{A_o(s)} \tag{2}$$

A corresponding state space description is determined by the 4-tuple $(\mathbf{A_o}, \mathbf{B_o}, \mathbf{C_o}, \mathbf{D_o})$. Assume that state feedback control is implemented using the control law

$$u(t) = -Kx(t) + \overline{r}(t) \tag{3}$$

Prove that

$$\frac{Y(s)}{\overline{R}(s)} = \frac{A_o(s)}{F(s)} G_o(s) \tag{4}$$

where $F(s) = \det(s\mathbf{I} - \mathbf{A_o} + \mathbf{B_o}\mathbf{K})$.

Solutions to Solved Problem 18.2

Solved Problem 18.3. *A system has transfer function given by*

$$G_o(s) = \frac{3}{(s+1)(s+2)(s+3)} = \frac{3}{s^3+6s^2+11s+6} \tag{5}$$

Choose state variables as follows:

$$x_1(t) = y(t) \tag{6}$$

$$x_2(t) = \frac{dy(t)}{dt} \tag{7}$$

$$x_3(t) = \frac{d^2y(t)}{dt^2} \tag{8}$$

18.3.1 *Find the state feedback gain $\mathbf{K} = [k_1\ k_2\ k_3]$ so as to achieve closed-loop poles located at -3, -4 and -5.*

18.3.2 *Say that the measuring device for the state $x_2(t)$ becomes faulty. Thus, k_2 might be multiplied for an unknown constant having any value in the range of $[0; 2]$. Will the closed loop remain stable?*

Solutions to Solved Problem 18.3

Solved Problem 18.4. *A discrete time signal has the form*

$$\eta[k] = A_1 \cos(\theta k) + A_2 \sin(\theta k) \tag{9}$$

where the frequency θ is known, but A_1 and A_2 are unknown.
Assume that $\eta[k]$ is measured (with some additive measurement noise, $w[k]$)
Show that A_1 and A_2 can be estimated using an observer based on the noisy measurement

$$y[k] = \eta[k] + w[k] \tag{10}$$

Solutions to Solved Problem 18.4

Solved Problem 18.5. *A discrete time system with input $u[k]$ and output $y[k]$ has transfer function given by*

$$G_q(z) = \frac{0.5z - 0.1}{(z - 0.4)(z - 0.8)} \tag{11}$$

18.5.1 *Build a state space model and design a state observer such that the estimation error settles to zero in two time units.*

18.5.2 *Test your design, by setting $u[k]$ equal to a sine wave of frequency $\pi/6$ and by adding a noise to the measurement of $y[k]$.*

Solutions to Solved Problem 18.5

Solved Problem 18.6. *Consider a system with transfer function*

$$G_o(s) = \frac{s + 3}{(s + 1)(s + 2)} \tag{12}$$

18.6.1 *Design an observer with poles located at $s = -6 \pm j4$.*

18.6.2 *Calculate a state estimate feedback gain to set the feedback poles to $s = -3$ and $s = -4$.*

18.6.3 *Combine the observer and state estimate feedback and determine the equivalent one degree of freedom controller, $C(s)$.*

18.6.4 *Compute the complementary sensitivity and explain why it has only three poles (instead of four).*

Solutions to Solved Problem 18.6

Chapter 18 - Solutions to Solved Problems

Solution 18.1. *To obtain the state state description we use the results in section §17.5 of the book and, in particular, equations (17.5.3) and (17.5.4) of the book. This yields*

$$\mathbf{A_o} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -5 \end{bmatrix}; \quad \mathbf{B_o} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad \mathbf{C_o} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}^T; \quad \mathbf{D_o} = 0 \tag{13}$$

The state feedback gain must satisfy

$$\det(s\mathbf{I} - \mathbf{A_o} + \mathbf{B_o}\mathbf{K}) = (s+4)(s+2)(s+3) = s^3 + 9s^2 + 26s + 24 \tag{14}$$

Since $\mathbf{K} = [k_1 \; k_2 \; k_3]$, (14) *can be written as*

$$\det(s\mathbf{I} - \mathbf{A_o} + \mathbf{B_o}\mathbf{K}) = \det \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -6 + k_1 & 1 + k_2 & s + 5 + k_3 \end{bmatrix} \tag{15}$$
$$= s^3 + (5 + k_3)s^2 + (1 + k_2)s - 6 + k_1 = s^3 + 9s^2 + 26s + 24$$

This leads to

$$\mathbf{K} = \begin{bmatrix} 30 & 25 & 4 \end{bmatrix} \tag{16}$$

Solution 18.2. *From equation (18.2.15) and (18.2.27) in the book, we have that*

$$\frac{Y(s)}{\overline{R}(s)} = \frac{\mathbf{C_o} Adj\{s\mathbf{I} - \mathbf{A_o} + \mathbf{B_o}\mathbf{K}\}\mathbf{B_o}}{F(s)} = \frac{\mathbf{C_o} Adj\{s\mathbf{I} - \mathbf{A_o}\}\mathbf{B_o}}{F(s)} \tag{17}$$

We also know that

$$G_o(s) = \frac{\mathbf{C_o} Adj\{s\mathbf{I} - \mathbf{A_o}\}\mathbf{B_o}}{\det(s\mathbf{I} - \mathbf{A_o})} \tag{18}$$

The result is obtained if we then multiply and divide equation (17) *by* $\det(s\mathbf{I} - \mathbf{A_o})$

Solution 18.3.

18.3.1 *We first obtain the 4-tuple* $(\mathbf{A_o}, \mathbf{B_o}, \mathbf{C_o}, \mathbf{D_o})$ *for the choice of state variables specified in the question.*

This leads to

$$\mathbf{A_o} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}; \quad \mathbf{B_o} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}; \quad \mathbf{C_o} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T; \quad \mathbf{D_o} = 0 \tag{19}$$

We next use the MATLAB command ***place*** *to find* $\mathbf{K}$*, using the following MATLAB code*

```
>>Go=tf(3,[1 6 11 6]); P=[-3 -4 -5]';
>>Ao=[0 1 0;0 1 0;-6 -11 -6];Bo=[0;0;3];
>>K=place(Ao,Bo,P);
```

This yields $\mathbf{K} = [18 \;\; 12 \;\; 2]$.

18.3.2 *We next consider the possibility of sensor measurement failure. We express* $\mathbf{K} = [18 \; k_2 \; 2]$*, where* $k_2 \in [0, 24]$

The closed-loop polynomial is given by

$$\det\left(s\mathbf{I} - \mathbf{A_o} + \mathbf{B_o}\mathbf{K}\right) = s^3 + 12s^2 + \lambda s + 60 \tag{20}$$

Where $\lambda \stackrel{\triangle}{=} 11 + 3k_2$*, hence* $\lambda \in [11, \;\; 83]$*. The effect of* λ *on the location of the closed-loop poles can be studied using a root locus approach for the polynomial in (20). The roots of this polynomial are the roots of*

$$1 + \lambda \frac{s}{s^3 + 12s^2 + 60} = 0 \tag{21}$$

We thus use the following MATLAB code

```
>>F=tf([1 0],[1 12 0 60]);lambda=[11:0.004:83];
>>rlocus(F,lambda);
```

The root locus is shown in Figure 1: We can see from this plot that the system is stable for all positive λ*. However it can be also appreciated that when the measurement has zero output (*λ *attains its minimal value) the dominant closed loop poles are close to the imaginary axis, yielding poorly damped natural modes.*

Solution 18.4. *We first note that the generating polynomial*[1]*,* $\Gamma_q(z)$*, for* $\eta[k]$ *is given by the denominator of the Z-transform of* $\eta[k]$*, i.e.,*

$$\Gamma_q(z) = z^2 - 2\cos(\theta)z + 1 \tag{22}$$

Hence the signal $\eta[k]$*is generated by the following (homogeneous) difference equation with (unknown) initial conditions*[2]*.*

$$\eta[k+2] - 2\cos(\theta)\eta[k+1] + \eta[k] = 0 \tag{23}$$

The measurement is given by (10).

We next define state variables as follows:

[1] See section §10.2 of the book

[2] Note that A_1 and A_2 depend on the initial conditions.

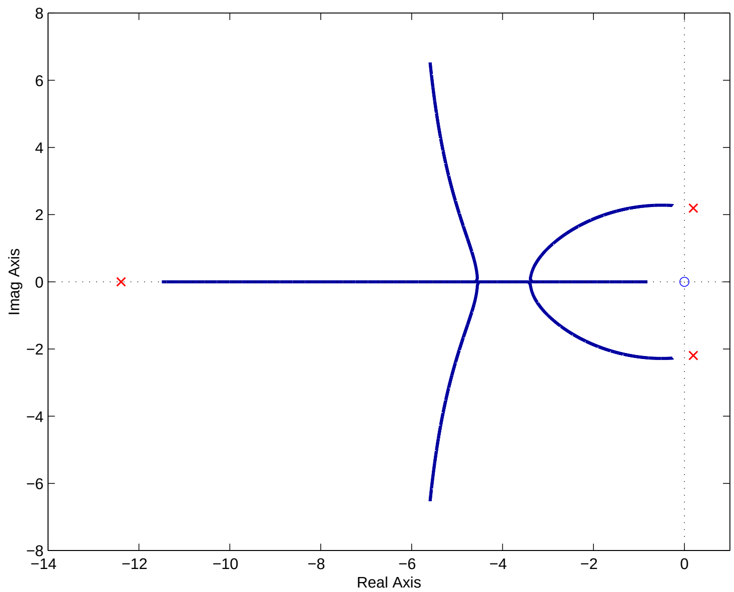

Figure 1: Root Locus for varying λ

$$x_1[k] = \eta[k] \tag{24}$$

$$x_2[k] = \eta[k+1] \tag{25}$$

The state space description for the signal is thus given by

$$\mathbf{x}[\mathbf{k}] \triangleq \begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2\cos(\theta) \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} \tag{26}$$

$$y[k] = [1 \;\; 0]\mathbf{x}[\mathbf{k}] + w[k] \tag{27}$$

We can thus build a state observer with gain $\mathbf{J_q}$ *to obtain estimates,* $\hat{x}_1[k]$ *and* $\hat{x}_2[k]$ *for* $x_1[k]$ *and* $x_2[k]$ *respectively.*

On the other hand, we note that

$$x_1[k] = A_1\cos(\theta k) + A_2\sin(\theta k) \tag{28}$$

$$x_2[k] = x_1[k+1] = (A_1\cos(\theta) + A_2\sin(\theta))\cos(\theta k) + (A_2\cos(\theta) - A_1\sin(\theta))\sin(\theta k) \tag{29}$$

$$= x_1[k]\cos(\theta) + A_2\sin(\theta)\cos(\theta k) - A_1\sin(\theta)\sin(\theta k) \tag{30}$$

We define

$$p\,[k] \triangleq \frac{x_2[k] - x_1[k]\cos(\theta)}{\sin(\theta)} = A_2\cos(\theta k) - A_1\sin(\theta k) \tag{31}$$

$$\tag{32}$$

If we knew the state, we could obtain A_1 *and* A_2 *from the equations for* $x_1[k]$ *and* $p\,[k]$ *as follows*

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{bmatrix} -\sin(\theta k) & \cos(\theta k) \\ \cos(\theta k) & \sin(\theta k) \end{bmatrix}^{-1} \begin{bmatrix} p\,[k] \\ x_1[k] \end{bmatrix} = \begin{bmatrix} -\sin(\theta k) & \cos(\theta k) \\ \cos(\theta k) & \sin(\theta k) \end{bmatrix} \begin{bmatrix} p\,[k] \\ x_1[k] \end{bmatrix} \tag{33}$$

However since we can only estimate the state, the best we can do is to estimate A_1 *and* A_2 *by* $\hat{A}_1$ *and* $\hat{A}_2$*, respectively, using the observed states* $\hat{x}_1[k]$ *and* $\hat{x}_2[k]]$*, as*

$$\begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} = \begin{bmatrix} -\sin(\theta k) & \cos(\theta k) \\ \cos(\theta k) & \sin(\theta k) \end{bmatrix} \begin{bmatrix} \hat{p}\,[k] \\ \hat{x}_1[k] \end{bmatrix} \tag{34}$$

where

$$\hat{p}[k] = \frac{\hat{x}_2[k] - \hat{x}_1[k]\cos(\theta)}{\sin(\theta)} \tag{35}$$

Solution 18.5.

18.5.1 *We follow the approach described in Example 17.4 of the book and the general formulae (17.5.11)-(17.5.12) of the book. We have that*

$$X_1(z) = \frac{1}{(z-0.4)(z-0.8)}U(z) \tag{36}$$

$$X_2(z) = zX_1(z) = \frac{z}{(z-0.4)(z-0.8)}U_q(z) \tag{37}$$

Thus

$$\mathbf{A_{oq}} = \begin{bmatrix} 0 & 1 \\ -0.32 & 1.2 \end{bmatrix}; \quad \mathbf{B_{oq}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \mathbf{C_{oq}} = \begin{bmatrix} -0.1 \\ 0.5 \end{bmatrix}^T; \quad \mathbf{D_{oq}} = 0 \tag{38}$$

For the estimation error to settle to zero in two time units, we need that the two observer poles are located at the origin.

The observer gain, $\mathbf{J_q}$*, can be found using MATLAB*[3]

```
>>Aoq=[0 1;-0.32 1.2];C=[-0.1 0.5];
>>Jq=acker(Aoq',Coq',[0 0.0])' ;
```

This yields $\mathbf{J_q} = [4/3 \;\; 8/3]^T$.

18.5.2 *To assess the effect of measurement noise we use the SIMULINK schematic in Figure 2.*

To build the simulation we have used the expressions in subsection §18.5.1 of the book, in particular expressions (18.5.2)-(18.5.4) of the book. The estimation error has been computed for the cases with and without noise. The plant input is taken to be a unit amplitude sine wave of frequency $\pi/6$*, and the noise was simulated as a uniform distributed random variable in* $(-0.25; 0.25)$*. To make the*

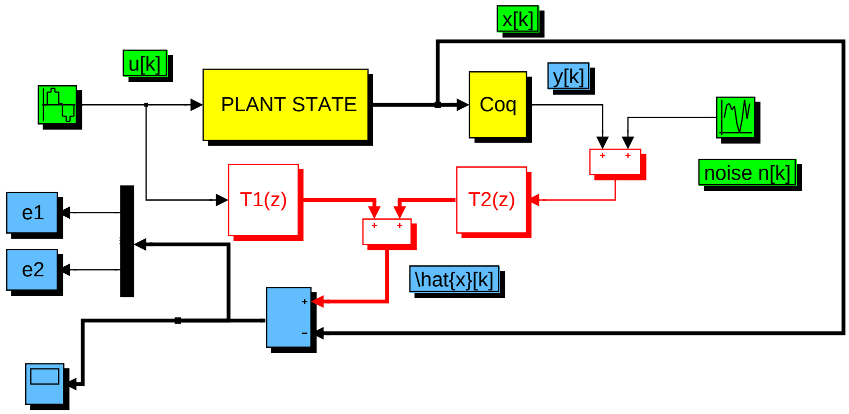

Figure 2: Observer with measurement noise

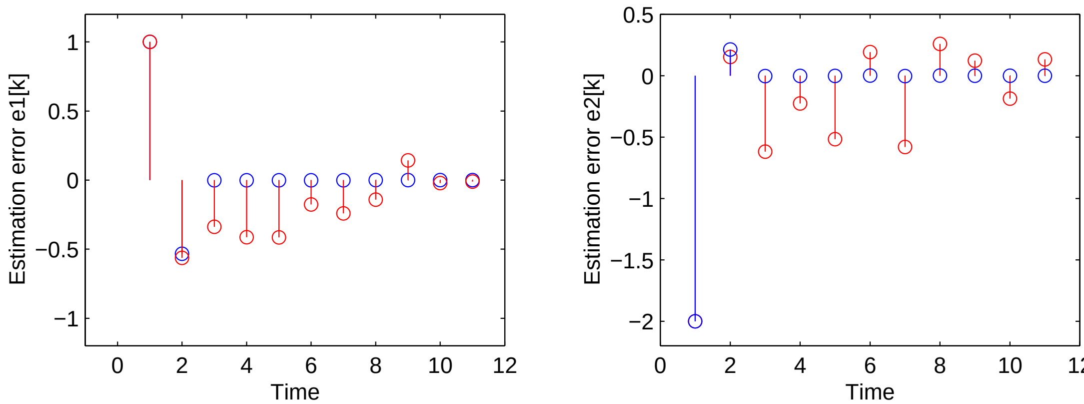

Figure 3: Estimation errors with noise(red) and without noise(blue)

simulation sensible we set a nonzero initial state on the plant, $(-1; 2)$. *The results are shown in Figure 3. The estimation errors are* $e_1[k]$ *and* $e_2[k]$.

We observe that in the noise-free case the estimation errors go to zero in two time units. However, in the presence of noise, the estimation errors will never converge. Indeed, because the observer has very high gain, the response is very sensitivity to measurement noise.

Solution 18.6. *We first find a state space description. We use MATLAB*

```
>>Go=tf([1 3],[1 3 2])
>>[Ao,Bo,Co,Do]=ssdata(Go);
```

This leads to

[3] MATLAB command **place** does not work for this choice of observer poles. Try it.

$$\mathbf{A_o} = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}; \quad \mathbf{B_o} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}; \quad \mathbf{C_o} = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}^T; \quad \mathbf{D_o} = 0 \tag{39}$$

18.6.1 *With the above state description, we compute the observer gain* $\mathbf{J}$ *through*

```
>> J=place(Ao',Co',[-6+i*4 -6-4*i])';
```

leading to $\mathbf{J} = [87 \quad -46]^T$

18.6.2 *To compute the state estimate feedback gain,* $\mathbf{K}$*, we use*

```
>> K=place(Ao,B,[-3 -4 ]);
```

leading to $\mathbf{K} = [2 \;\; 2.5]$.

18.6.3 *The equivalent one degree of freedom controller is given by the 4-tuple*[4] $(A_o - B_oK - JC_o, J, K, 0)$.

This leads to a controller given by equation (18.5.14) of the book, which can be computed with the MATLAB command ***ss2tf***

```
>>[numC,denC]=ss2tf(Ao-Bo*K-J*Co,J,Co,0);C=tf(numC, denC)
```

This leads to

$$C(s) = \frac{59s + 182}{s^2 + 16s + 39} = \frac{59s + 182}{(s + 3)(s + 13)} \tag{40}$$

18.6.4 *The complementary sensitivity can be computed with MATLAB yielding*

$$T_o(s) = \frac{59s + 182}{s^3 + 16s^2 + 100s + 208} = \frac{59s + 182}{(s + 4)(s + 6 + j4)(s + 6 - j4)} \tag{41}$$

Thus, $T_o(s)$ *has only three poles. However, the general form of the closed loop characteristic polynomial is* $A_{cl}(s) = E(s)F(s)$*, where* $E(s)$ *is the observer polynomial and* $F(s)$ *is the state feedback polynomial. The key issue here is that we have chosen* $F(s) = (s + 3)(s + 4)$ *while the plant has also a zero at* $s = -3$*. Therefore, there must be a pole-zero cancellation; this can be seen in the denominator of* $C(s)$ *of equation* (40).

[4] See subsection §18.5.2 of the book.