Chapter 15 - Solved Problems

Solved Problem 15.1. *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*

Consider a plant having a nominal model given by

$$G_o(s) = \frac{1}{s+2} \tag{1}$$

The aim of the control design is to follow a constant reference. Also, an input disturbance has significant energy in the frequency band $[0, 6][rad/s]$. Find a proper controller using the affine parameterization.

Solutions to Solved Problem 15.1

Solved Problem 15.2. *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*

Consider a plant having nominal model given by

$$G_o(s) = \frac{s-2}{(s+1)(s+3)} \tag{2}$$

Assume that the bandwidth of the closed loop has to be limited to $8[rad/s]$ in order to have good noise immunity. We further assume that the reference is constant. Design a controller using the affine parameterization.

Solutions to Solved Problem 15.2

Solved Problem 15.3. *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*

Consider a plant having nominal model given by

$$G_o(s) = \frac{1}{s^2 + 1.4s + 1} \tag{3}$$

Find a PID controller that stabilizes the plant and provides zero steady state error at zero frequency.

Solutions to Solved Problem 15.3

Solved Problem 15.4. *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*

Consider a plant having nominal model given by

$$G_o(s) = \frac{e^{-s}}{(s+1)(s+2)} \tag{4}$$

using the Smith controller of Figure 15.5 of the book, find a controller that stabilizes the plant and allows good tracking (although delayed!!) of a reference with energy in the band $[\,0, 2]$ $[rad/s]$.

Solutions to Solved Problem 15.4

Solved Problem 15.5. *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*

Consider a plant having nominal model given by

$$G_o(s) = \frac{1}{s-1} \tag{5}$$

Find a controller that stabilizes the plant, ensures perfect reference tracking at zero frequency and provides good compensation of disturbances with energy in the frequency band $[0, 1]$ $[rad/s]$.

Solutions to Solved Problem 15.5

Solved Problem 15.6. *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*

Consider a plant having nominal model given by

$$G_o(s) = \frac{1}{s-1} \tag{6}$$

Find all stabilizing controllers for this plant.

Solutions to Solved Problem 15.6

Solved Problem 15.7. *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*

Show that the controller found in Problem 15.5 is consistent with the solution to Problem 15.6

Solutions to Solved Problem 15.7

Solved Problem 15.8. *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*

Consider a discrete time plant having nominal transfer function given by

$$G_{oq}(z) = \frac{z+2}{(z-0.9)(z-0.1)} \tag{7}$$

Find a controller that stabilizes the plant, ensure perfect tracking of a constant reference and exhibits a closed-loop step response faster than $(0.7)^k$.

Solutions to Solved Problem 15.8

Solved Problem 15.9. *Contributed by - Alvaro Liendo, Universidad Tecnica Federico Santa Maria, Chile.*

Consider a discrete time plant having nominal transfer function given by

$$G_{oq}(z) = \frac{1}{(z+2)} \tag{8}$$

Find all controllers that stabilize the plant and ensure zero steady state error when tracking a constant reference.

Solutions to Solved Problem 15.9

Chapter 15 - Solutions to Solved Problems

Solution 15.1. *To achieve perfect tracking,* $T_o(0) = 1$ *is needed.*

Also, we have that

$$S_{io}(s) = [1 - T_o(s)]G_o(s) = \frac{1}{(s+2)}[1 - T_o(s)] \tag{9}$$

Then, to achieve good disturbance rejection, we need the bandwidth of $T_o(s)$ *to be at least* $6[rad/s]$ *Also* $[1 - T_o(s)]$ *should cancel the factor* $(s+2)$ *in* S_{io}*. This leads to* $T_o(-2) = 1$*. We choose*

$$Q(s) = \frac{36(\tau s + 1)(s+2)}{s^2 + 8s + 36} \tag{10}$$

The condition $T_o(-2) = Q(-2)G_o(-2) = 1$ *implies* $\tau = \frac{1}{6}$*. Thus*

$$Q(s) = \frac{6(s+6)(s+2)}{s^2 + 8s + 36} \tag{11}$$

The quality of the disturbance compensation can be evaluated by plotting the magnitude of the input sensitivity function. This is shown in Figure 1.

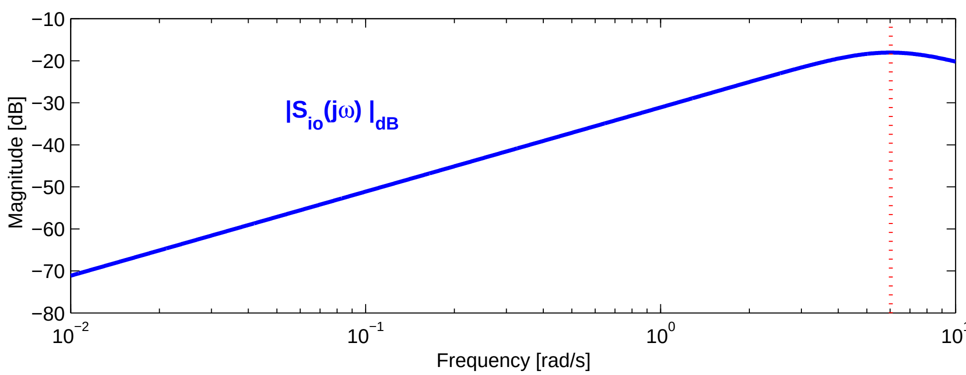

Figure 1: Input sensitivity frequency response

From Figure 1 we see that, in the disturbance frequency band, $[0, 6]$ $[rad/s]$*, the gain is at most* -17 $[dB]$.

Solution 15.2. *We know that* $(s-2)$ *must be a zero of* $T_o(s)$ *to ensure an internally stable closed loop. Also,* $T_o(0)$ *must be zero to ensure perfect steady state reference tracking. Also, to achieve the fastest possible loop, we set the poles in* $T_o(s)$ *close to* $8[rad/s]$*. Say we choose*

$$T_o(s) = \frac{-32(s-2)}{(s^2 + 12s + 64)} \tag{12}$$

This leads to

$$Q(s) = \frac{-32(s+1)(s+3)}{(s^2+12s+64)} \tag{13}$$

Solution 15.3. *We use the theory developed in subsection §15.4.3 of the book. To achieve a proper inverse we select*

$$Q(s) = F_Q(s)[G_o(s)]^{-1} = F_Q(s)(s^2+1.4s+1) \tag{14}$$

We choose a natural frequency of 2.5 and a damping factor of 0.65. Hence

$$F_Q(s) = \frac{1}{0.16s^2+0.52s+1} \tag{15}$$

And we then have

$$Q(s) = \frac{s^2+1.4s+1}{0.16s^2+0.52s+1} \tag{16}$$

Using equations (15.4.16) to (15.4.19) in the book we finally have

$$K_P = 2.10 \tag{17}$$
$$K_I = 1.92 \tag{18}$$
$$K_D = 0.13 \tag{19}$$
$$\tau_D = 0.31 \tag{20}$$

The closed-loop step response using this controller is shown in Figure 2.

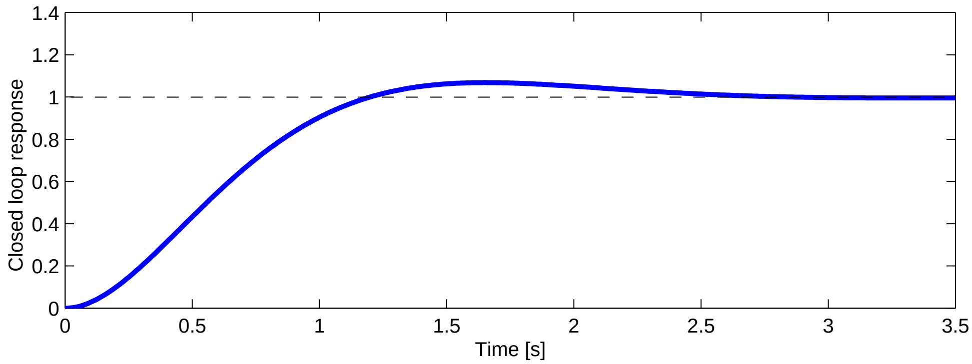

Figure 2: Closed-loop step response

Solution 15.4. *Using the structure of Figure 15.5 in the book, we find a controller for the rational part of* $G_o(s)$*, denoted by* $\overline{G}_o(s)$*. An appropriate* $Q(s)$ *(achieving a closed-loop bandwidth larger than 2 [rad/s]) is*

$$Q(s) = \frac{20(s+1)(s+2)}{(s^2+2.8s+4)(s+10)} \tag{21}$$

leading to

$$T_o(s) = 40\frac{e^{-s}}{(s^2+2.8s+4)(s+10)} \tag{22}$$

Note that $T_o(0) = 1$*, i.e., the controller provides integral action.*

Solution 15.5. *To ensure internal stability of the closed loop it is necessary and sufficient that* $Q(s)$ *be stable and that* $(s-1)$ *be a zero of* $Q(s)$ *and of* $(1 - G_o(s)Q(s))$*. We choose*

$$Q(s) = (s-1)\overline{Q}(s) \tag{23}$$

where

$$\overline{Q}(s) = \frac{P(s)}{(s+2)(s+3)} \tag{24}$$

with $P(s)$ *a polynomial satisfying* $P(0) = 6$*, to achieve zero steady state error for constant references. (This forces* $\overline{Q}(0) = 1$*.)*

Furthermore,

$$1 - G_o(s)Q(s) = 1 - \overline{Q}(s) = \frac{s^2+5s+6-P(s)}{(s+2)(s+3)} \tag{25}$$

must have a zero at $s = 1$ *and one other zero. This leads to*

$$s^2 + 5s + 6 - P(s) = (s-1)(s+\alpha) \tag{26}$$

$$P(s) = (6-\alpha)s + (6+\alpha) \tag{27}$$

To satisfy $P(0) = 6$*, we choose* $\alpha = 0$*. This leads to*

$$\overline{Q}(s) = \frac{6s+6}{(s+2)(s+3)} \tag{28}$$

$$Q(s) = \frac{6(s+1)(s-1)}{(s+2)(s+3)} \tag{29}$$

With this choice we have that

$$T_o(s) = Q(s)G_o(s) = \frac{6(s+1)}{(s+2)(s+3)} \tag{30}$$

Note that $T_o(s)$ *satisfies the interpolation constraint* $T_o(1) = 1$*.*

Solution 15.6. *We know that all controllers that stabilize this plant have the form (see equation (15.7.11) in the book)*

$$Q(s) = \frac{s-1}{F(s)}\left[\frac{P(s)}{E(s)} + Q_u(s)\frac{(s-1)}{E(s)}\right] \tag{31}$$

where $Q_u(s)$ is any stable transfer function; $F(s)$, $E(s)$ are stable polynomials and $P(s)$, $F(s)$, $E(s)$ satisfy the pole assignment equation

$$(s-1)L(s) + P(s) = E(s)F(s) \tag{32}$$

$$sL(s) + (P(s) - L(s)) = E(s)F(s) \tag{33}$$

Note that $P(s)/L(s)$ is any stabilizing controller for this plant. Choosing $L(s) = 1$, $P(s) = 2$ and $E(s) = 1$ we have $F(s) = s+1$ This leads to

$$Q(s) = \frac{(s-1)}{(s+1)}[2 + (s-1)Q_u(s)] \tag{34}$$

Solution 15.7. *We only need to find the $Q_u(s)$ in Problem 15.6 that leads to $Q(s)$ in Problem 15.5 and verify that is stable*

$$\frac{(s-1)}{(s+1)}[2 + (s-1)Q_u(s)] = \frac{6(s+1)(s-1)}{(s+2)(s+3)} \tag{35}$$

After some algebra, we find

$$Q_u(s) = \frac{4s+6}{(s+2)(s+3)} \tag{36}$$

This shows consistency because $Q_u(s)$ is found to be stable as required in the solution to Problem 15.6.

Solution 15.8. *To have a stable $Q_q(z)$, $T_{oq}(z)$ must contain the unstable zero of $G_{oq}(z)$. Also, to follow a constant reference we need $T_{oq}(1) = 1$. Finally, to ensure that the closed-loop step response has transients faster than $(0.7)^k$, $T_{oq}(z)$ must contain only poles inside the disk with radius 0.7. This requires that $Q_q(z)$ cancels the slow plant pole (located at $z = 0.9$). Considering this, we choose*

$$T_{oq}(z) = K\frac{z+2}{(z-0.1)^2} \tag{37}$$

The condition $T_{oq}(1) = 1$ leads to $K = 0.27$ and

$$T_{oq}(z) = 0.27\frac{z+2}{(z-0.1)^2} \tag{38}$$

This yields

$$Q_q(z) = 0.27\frac{(z-0.9)}{(z-0.1)} \tag{39}$$

Solution 15.9. *We first find all stabilizing controllers for this plant. They have the form (see equation (15.7.11) in the book)*

$$Q_q(z) = \frac{z+2}{F_q(z)}\left[\frac{P_q(z)}{E_q(z)} + Q_{uq}(z)\frac{(z+2)}{E_q(z)}\right] \tag{40}$$

where $Q_{uq}(z)$ *is any stable transfer function;* $F_q(z)$, $E_q(z)$ *are stable polynomials and* $P_q(z)$, $F_q(z)$, $E_q(z)$ *satisfy the pole assignment equation*

$$(z+2)L_q(z) + P_q(z) = E_q(z)F_q(z) \tag{41}$$

$$zL_q(z) + [P_q(z) + 2L_q(z)] = E_q(z)F_q(z) \tag{42}$$

Note that $P_q(z)/L_q(z)$ *is any stabilizing controller for this plant. Choosing* $L_q(z) = 1$, $P_q(z) = -2.5$ *and* $E_q(z) = 1$ *we have* $F_q(z) = z - 0.5$ *This leads to*

$$Q_q(z) = \frac{(z+2)}{(z-0.5)}[-2.5 + (z+2)Q_{uq}(z)] \tag{43}$$

This parameterizes all controllers that stabilize the plant. We next need to ensure perfect reference tracking. This is achieved by setting $Q_q(1) = [G_{oq}(z)]^{-1} = 3$*. Hence, we require*

$$Q_q(1) = 6[-2.5 + 3Q_{uq}(1)] = 3 \tag{44}$$

This leads to $Q_{uq}(1) = 1$*. The only other requirement is that* $Q_uq(z)$ *must be stable.*