Chapter 14 - Solved Problems

Solved Problem 14.1.

14.1.1 *Consider the digital control of a simple double integrator system using a zero order hold. Investigate the associated "sampling zero" using both*

- **(i)** *The algebraic expression for the model, and*
- **(ii)** *the Poisson summation formula in the frequency domain.*

14.1.2 *Show by a similar argument that all systems having relative degree $2k$ (with k a positive integer), will have a zero at $z = -1$ with fast sampling.*

Solutions to Solved Problem 14.1

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Solution 14.1.

14.1.1

(i) *From equation (12.13.4) in the book, the discrete transfer function is*

$$\begin{aligned} H_q(z) &= (1 - z^{-1}) Z\{\textit{Sampled step response of } G_o(s)\} \\ &= (1 - z^{-1}) Z\{\textit{Sampled step response of } \frac{1}{s^2}\} \\ &= (1 - z^{-1}) Z\{\textit{Samples of } \frac{t^2}{2}\} \\ &= (1 - z^{-1}) Z\{\frac{(k\Delta)^2}{2}\} \end{aligned} \tag{1}$$

Using Table 12.1 in the book, we have

$$\begin{aligned} Hq(z) &= (1 - z^{-1}) \frac{\Delta^2}{2} \left(\frac{z(z+1)}{(z-1)^3} \right) \\ &= \frac{\Delta^2}{2} \frac{(z+1)}{(z-1)^2} \end{aligned} \tag{2}$$

We thus see that the sampled data model has a zero at $z = -1$ whereas the continuous time model has no finite zero. Thus, $z = -1$ is a sampling zero.

(ii) *We see from part (i) that the sampling zero appears in the z-transform of the sampled step response of $\frac{1}{s^2}$. From §14.6 of the book, the Poisson summation formula gives the discrete frequency response as (note: sampling in the time domain ⇒ folding in the frequency domain):*

$$F_q(e^{j\omega\Delta}) = \frac{1}{\Delta} \sum_{k=-\infty}^{\infty} F(j\omega + jk\frac{2\pi}{\Delta}) + \frac{f(0)}{2} \tag{3}$$

Now $f(0) = 0$. Also, the Fourier Transform of the continuous step response is

$$\begin{aligned} F(j\omega) &= \frac{1}{(j\omega)^3} \\ &= \frac{j}{\omega^3} \end{aligned} \tag{4}$$

From equation (3), we see that the discrete frequency response at $\omega = \frac{\pi}{\Delta}$ is $\frac{1}{\Delta}\left(F\frac{j\pi}{\Delta} + F\frac{-j\pi}{\Delta}\right) + \frac{1}{\Delta}\left(F\frac{j2\pi}{\Delta} + F\frac{-j2\pi}{\Delta}\right) + \dots$. However, using equation (4), we see that every term in the above sequence is zero. That is, folding leads to cancellation which produces zero discrete frequency response at $\frac{\pi}{\Delta}$. This confirms the presence of the zero at $z = -1$.
Finally note that the above conclusion would hold equally well for any continuous time model of relative degree 2 provided a fast sampling rate was utilized. The reason is that, for Δ small, $\omega = \frac{\pi}{\Delta}$ will be large and hence the system will behave like $\frac{C}{s^2}$ in the vicinity of $\omega = \frac{\pi}{\Delta}$.

14.1.2 *With fast sampling, the system will behave like* $\frac{c}{s^m}$*;* $m = 2k$*. The step response then behaves like* $\frac{c}{s^{m+1}}$*. This gives the Fourier Transform of the step response as*

$$F(j\omega) = \frac{c(-1)^k}{j\omega^{m+1}} \tag{5}$$

An argument, similar to that used in 14.1.1, based on folding then gives $F_q(e^{j\pi}) = 0$ *corresponding to a zero at* $z = -1$*.*
