Chapter 9

Frequency Domain Design Limitations

The purpose of this chapter is to develop frequency domain constraints and to explore their interpretations. The results to be presented here have a long and rich history beginning with the seminal work of Bode published in his 1945 book on network synthesis. The results give an alternative view of the fundamental time domain limitations presented in Chapter 8.

Some History

H. Bode, *Network Analysis and Feedback Amplifier Design*, Van Nostrand, 1945. (*Work done 1938, 1939*). (*Based on course at Bell Telephone Laboratories on Feedback Amplifier Design for Long Distance Telephone systems.*)

Typical results:

(1) For *Stable Min. phase Transfer Functions* gain can be computed from phase and vice versa, e.g.

$$\phi = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dA}{du}\log\coth\frac{|u|}{2}du$$

(2) *Log sensitivity trade-off* (sometimes called *Water bed effect*)

$$\int_{0}^{\infty}\log|S|dw = 0$$

For historical interest we include (*as the next few slides*) the first few pages of the book written by Bode

Network Analysis and Feedback Amplifier Design

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PREFACE

This book was originally written as an informal mimeographed text for one of the so-called "Out-of-Hour" courses at Bell Telephone Laboratories. The bulk of the material was prepared in 1938 and 1939 and was given in course form to my colleagues there in the winters of 1939–40 and 1940–41. During the war, however, the text has also been supplied as a reference work to a considerable number of other laboratories engaged in war research. The demand for the text on this basis was unexpectedly heavy and quickly exhausted the original supply of mimeographed copies. It has consequently been decided to make the text more widely available through regular channels of publication.

In revising the material for publication, the original theoretical discussion has been supplemented by footnote references to other books and papers appearing both before and after the text was first written. In addition, an effort has been made to simplify the theoretical treatment in Chapter IV, and minor editorial changes have been made at a number of points elsewhere. Otherwise, however, the text is as it was originally written.

The book was first planned as a text exclusively on the design of feedback amplifiers. It shortly became apparent, however, that an extensive preliminary development of electrical network theory would be necessary before the feedback problem could be discussed satisfactorily. With the addition of other logically related chapters, this has made the book primarily a treatise on general network theory. The feedback problem is still conspicuous, but the book also contains material on the design of non-feedback as well as feedback amplifiers, particularly those of wide band type, and on miscellaneous transmission problems arising in wide band systems generally. Much of this is material which has not hitherto appeared in previous texts on network theory. On the other hand, transmission line and filter theory, which are the primary concerns of most earlier network texts, are omitted.

Two further explanatory remarks may be helpful in understanding the book. The first is the fact that, although the feedback amplifiers envisaged in most of the discussion are of the conventional single loop, absolutely stable type, the original plan for the text called for two final chapters on design methods appropriate for multiple loop and conditionally stable circuits. Invincible fatigue set in before these chapters could the pole at the origin. Then evidently the total central angle subtended by all four sides would be zero, so that the loop integral would vanish.

An example of a different sort is furnished by one of the classical theorems in the calculus of residues. Let $g(z)$ be a function which is analytic on and within a given closed contour and let q be any point within the contour. Then $g(z)/(z-q)$ is a function which is analytic in the same region except for a simple pole at $z = q$. The residue at this pole must be $g(q)$, the value assumed by g when $z = q$, as we can easily see by expanding $g(z)$ near this point in the Taylor's series $g(q) + g'(q)(z-q) + (1/2!)g''(q)(z-q)^2 + \cdots$, and noticing that after division by $z - q$ the series takes the same form as that given for $f(z)$ in (8–8). If we identify $g(z)/(z-q)$ with $f(z)$, therefore, (8–9) allows us to write

$$\oint \frac{g(z)}{z-q}\,dz = -2\pi i g(q) \tag{8–13}$$

where, as before, the integration is taken in a clockwise direction.

This theorem is of interest here because of its bearing on the general problem of relating the values assumed by an analytic function within a given region to its values on the boundary of the region, which was discussed earlier in the chapter. Evidently, if we know $g(z)$ we can perform the integration on the left-hand side of (8–13) and calculate the special value $g(q)$ directly. In order to make this possible, however, we need know $g(z)$ only on the path of integration, that is, only on the boundary. Equation (8–13) thus provides a method of determining an analytic function anywhere inside a given region from a knowledge of its behavior only on the boundary of the region. The problem with which we are actually concerned is that of determining what properties a function must have on the boundary of the region when it is known to have certain properties in the interior. This problem is evidently in many respects the converse of that solved by (8–13), although it is much more general, since we begin with a specification only of the general properties of the function rather than with a knowledge of its behavior in detail. On this account it is not possible to present an adequate answer in terms of a single compact formula such as (8–13). The range of questions of practical interest requires the development of a considerable variety of formulae, only a few of which are given in the present chapter. Except for these qualifications, however, the solution of the converse problem will be found to imply relations between the values of a function on the boundary of a region and in its interior as tightly knit as that given by (8–13).

8.6. *Integral of the Logarithmic Derivative*

For the immediate purposes of the present chapter, the preceding disabruptly to zero, the feedback available over a definite 12.5 or 25 mc band is slightly less than these figures would indicate.*

13.6. *Regeneration and Degeneration in a General Feedback Circuit*

The analysis just concluded can also be used to derive a second result which is at least curious, although it may not be of great practical importance. We are accustomed to thinking of feedback amplifiers as being either regenerative, in which case the external gain is increased at the cost of an increase in the effects of tube variations, or degenerative, in which case the gain is reduced in exchange for a corresponding improvement in the effects of tube variations. It is apparent, however, that the expression $\log(1 + G_m Z)$, which we have just studied, is merely a particularly simple form to which the general expression $\log(1 - \mu\beta)$ reduces for the special case of a single tube feedback amplifier. Equation (13–15) therefore measures the total reduction in gain or degeneration for such a system. In a similar fashion we might replace the θ in (13–13) by $\log(1 - \mu\beta)$ for a general amplifier and proceed with the analysis in the same way as before.

In the general case only one difference would appear. Whereas in the single tube feedback amplifier the feedback $\mu\beta$ varies, in general, inversely as the first power of the frequency at high frequencies, in a general multitube amplifier, the feedback would vanish as some higher power of frequency. If the feedback drops off as a higher power than the first, however, the contribution of the integral around the infinite semicircle is evidently zero and the right-hand side of equation (13–15) therefore vanishes. This can be formulated as the

> *Theorem:* In a single loop feedback amplifier of more than one stage the average regeneration or degeneration over the complete frequency spectrum is zero.

In a typical amplifier, in other words, the increase in gain at high frequencies due to the fact that $|1 - \mu\beta|$ is less than one just balances the

* This example is taken from the design of a repeater amplifier used some years ago in an experimental system for long distance broad-band transmission over coaxial lines. The system was intended to transmit carrier telephone messages over a 2 mc band; a modified form of the system with a somewhat extended band to accommodate television as well as telephone signals is described by Strieby and Wentz, "Television Transmission over Wire Lines," *B.S.T.J.*, Jan., 1941. The 40 $\mu\mu f$ cathode-ground capacity mentioned in the text is much greater than the physical capacity in the actual amplifier, but the grid-cathode and plate-cathode capacities lead to an effective C of about this magnitude. The reason for maintaining the local feedback over a band as great as 12 to 25 mc is that otherwise the stability of the system is jeopardized by a decrease in the gain of the tube to which the local feedback is applied, even if the characteristics around the main loop are apparently absolutely stable. The design is described in more detail in a later chapter.

Theme

In Advanced Control, understanding what can and cannot be done (and why) is often more important than producing a specific design.

The constraints presented here mirror constraints that apply in many other areas, e.g.

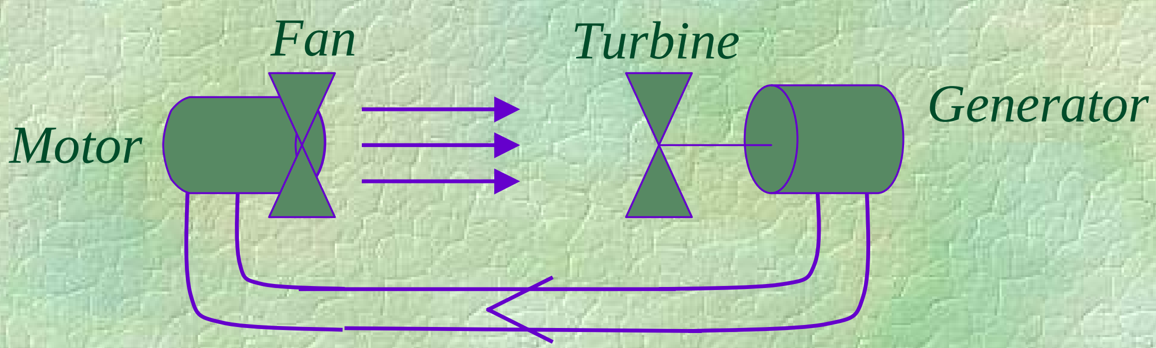
- *Second Law of Thermodynamics*
- *Cramer Rao Inequality of Estimation*

Design Constraints in Engineering

Examples: First and Second Laws of Thermodynamics

(a) It can rule out *silly* ideas:

For example, consider the following Perpetual Motion Machine?



(*Ruled out by fundamental principle of conservation of energy*)

(b) They can also quickly identify fundamentally hard problems:

For example, if faced with the following problem:

"*Design a coal-fired generating plant with 80% efficiency*"

This can be shown to be impractical using fundamental laws, e.g., using 80% efficiency and ideas of entropy we see that the required temperature is unrealistic, e.g.

$$\frac{T - T_a}{T} > 80\% \Rightarrow T > 5 \times T_a \Rightarrow T > 1227^\circ C!$$

Examples from other fields

- Estimation
  - Cramer Rao inequality

$$E\left\{\left(\hat{\theta}-\theta\right)^2\right\} \geq \left(E\left\{\frac{\partial \log p(y;\theta)}{\partial \theta}\right\}^2\right)^{-1}$$

- Communications
  - Probability of error can be made arbitrarily small provided

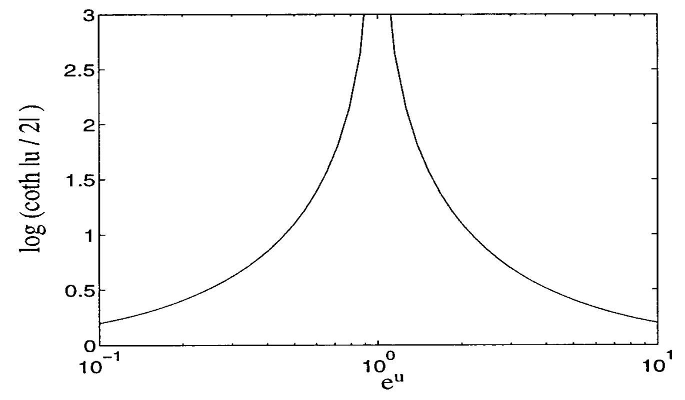
$$R \leq C$$

$$C = B \log_2\left(1+\frac{S}{N}\right) \text{bits / sec}$$

One of the best known results is that for a stable minimum phase system, the phase is uniquely determined by the magnitude and vice versa. The exact formula is given on the next slide.

Weighting Function

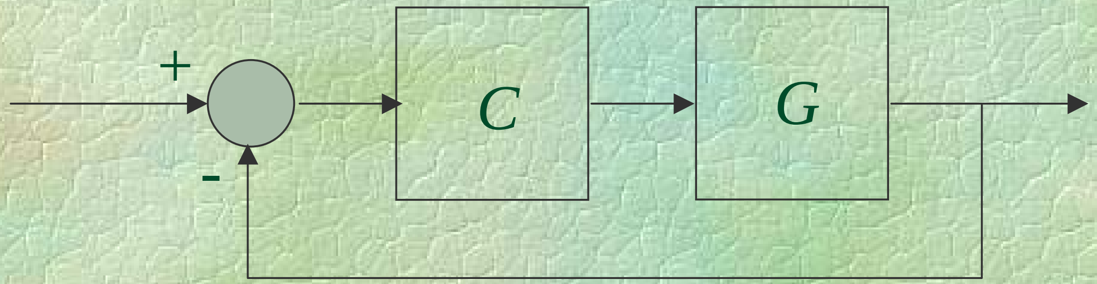
$$\phi(\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d \log |H(j\omega_0 e^u)|}{du} \log \coth \left|\frac{u}{2}\right|$$



Thus, the slope of the magnitude curve in the vicinity of ω_0, say c, determines the phase $\phi(\omega_0)$:

$$\phi(\omega_0) \approx \frac{c}{\pi}\int_{-\infty}^{\infty} \log \coth \left|\frac{u}{2}\right| du = \frac{c}{\pi}\frac{\pi^2}{2} = c\frac{\pi}{2}$$

Here we study the fundamental design constraints that apply to feedback systems of the type illustrated below



The constraints we develop apply to frequency domain integrals of the sensitivity function and complementary sensitivity function.

Conceptual Background

Before delving into technical details, we first review the conceptual nature of the results to be presented.

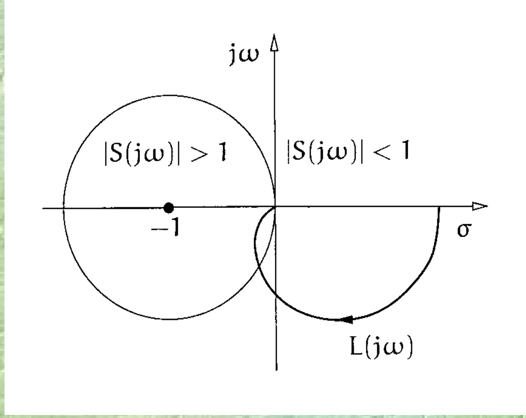
The simplest (*and perhaps best known*) result is that, for an open loop stable plant, the integral of the logarithm of the closed loop sensitivity is zero; i.e.

$$\int_0^\infty \ln |S_0(jw)| \, dw = 0$$

Now we know that the logarithm function has the property that it is negative if $|S_0| < 1$ and it is positive if $|S_0| > 1$.

Graphical interpretation of the area formula

Typical Nyquist plot for a stable rational transfer function, *L*, of relative degree two:



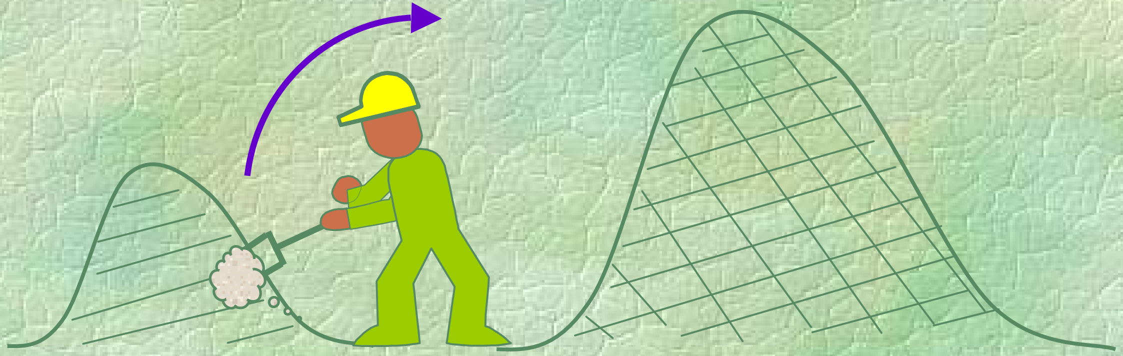
Note that $S(jT)^{-1} = 1 + L(j\omega)$ is the vector from the -1 point to the point on the Nyquist plot corresponding to the frequency ω.

It is clear from the plot that frequencies where |S(jw)| < 1 are balanced by frequencies where |S(jw)| > 1.

The above result implies that set of frequencies over which sensitivity reduction occurs (i.e. where $|S_0| < 1$) must be matched by a set of frequencies over which sensitivity magnification occurs (i.e. where $|S_0| > 1$).

This has been given a nice (*cartoon like*) interpretation as thinking of sensitivity as a pile of dirt. If we remove dirt from one set of frequencies, then it piles up at other frequencies. This is conceptually illustrated on the next slide.

Physical Interpretation



Sensitivity dirt

We will show in the sequel that the above sensitivity trade-off is actually made more difficult if there are either (*or both*)

- *Right half plane open loop zeros*
- *Right half plane open loop poles*

Notice that these results hold irrespective of how the control system is designed; i.e. they are fundamental constraints that apply to all feedback solutions.

Indeed, it will turn out that to avoid large frequency domain sensitivity peaks it is necessary to limit the range of sensitivity reduction to be:

(i) *less than any right half plane open loop zero*

(ii) *greater than any right half plane open loop pole*.

This begs the question - "*What happens if there is a right half plane open loop zero having smaller magnitude than a right half plan open loop pole?*"

Clearly the requirements specified on the previous slide are then mutually incompatible. The consequence is that large sensitivity peaks are unavoidable and, as a result, poor feedback performance is inevitable.

An example precisely illustrating this conclusion will be presented later.

We begin with Bode's integral constraint on sensitivity. This is a formal statement of the result discussed above at a conceptual level; namely

$$\int_0^\infty \ln |S_0(jw)|\, dw = 0$$

Bode's Integral Constraints on Sensitivity

Consider a one d.o.f. stable control loop with open loop transfer function

$$G_o(s)C(s) = e^{-s\tau} H_{ol}(s) \qquad \tau \geq 0$$

where $H_{0l}(s)$ is a rational transfer function of relative degree $n_r > 0$ and define

$$\kappa \stackrel{\triangle}{=} \lim_{s\to\infty} sH_{ol}(s)$$

Assume that $H_{0l}(s)$ has no open loop poles in the open RHP. Then the nominal sensitivity function satisfies:

$$\int_0^\infty \ln|S_o(j\omega)| d\omega = \begin{cases} 0 & \text{for } \tau > 0 \\ -\kappa \frac{\pi}{2} & \text{for } \tau = 0 \end{cases}$$

Graphically, the above statement can be appreciated in Figure 9.1

In Figure 9.1, the area A1 ($\Rightarrow |S_0(jw)| < 1$) must be equal to the area A2 ($\Rightarrow |S_0(jw)| > 1$), to ensure that the integral in equation (9.2.3) is equal to zero.

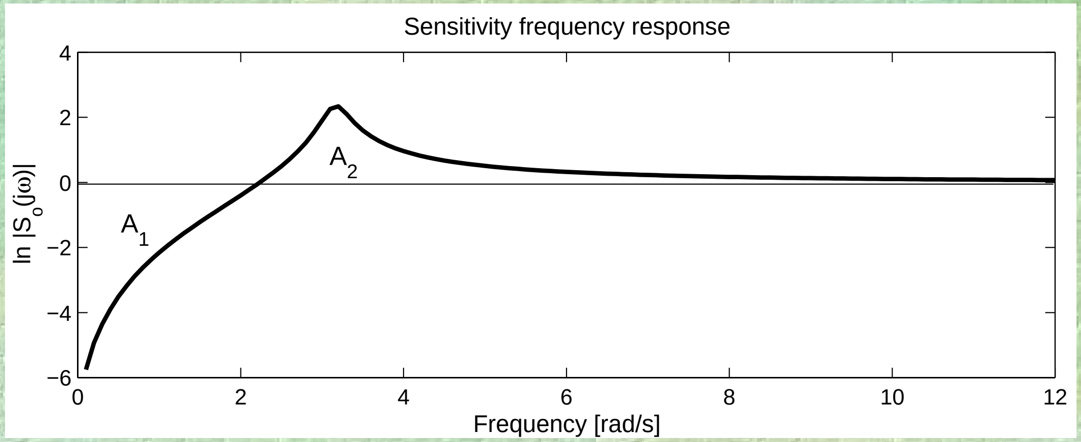


Figure 9.1: *Graphical interpretation of the Bode integral*

Proof

We will not give a formal proof of this result. Suffice to say it is an elementary consequence of the well known Cauchy integral theorem of complex variable theory - see the book for details.

The extension to *open loop unstable systems* is as follows:

Consider a feedback control look with open loop transfer function as in Lemma 9.1, and having unstable poles located at $p_1, \ldots, p_N$, pure time delay τ, and relative degree $n_r \geq 1$. Then, the nominal sensitivity satisfies:

$$\int_0^\infty \ln|S_o(j\omega)|\,d\omega = \pi \sum_{i=1}^{N} \mathcal{R}\{p_i\} \qquad \text{for} \quad n_r > 1$$

$$\int_0^\infty \ln|S_o(j\omega)|\,d\omega = -\kappa\frac{\pi}{2} + \pi \sum_{i=1}^{N} \mathcal{R}\{p_i\} \qquad \text{for} \quad n_r = 1$$

where $\kappa = \lim_{s\to\infty} sH_{0l}(s)$

Observations

We see from the above results that with open loop RHP poles, the integral of log sensitivity is required to be greater than zero (*previously*) it had to be zero. This makes sensitivity minimization more difficult.

Design Interpretations

Several factors such as undermodeling, sensor noise, plant bandwidth, etc., lead to the need of setting a limit on the bandwidth of the closed loop. Typically

$$|L(j\omega)| \leq \delta \left(\frac{\omega_c}{\omega} \right)^{1+k}, \qquad \forall \omega \geq \omega_c,$$

where $\delta < 1/2$ and $k > 0$.

Corollary: Suppose that L is a rational function of relative degree two or more and satisfying the bandwidth restriction. Then

$$\left| \int_{\omega_c}^{\infty} \log |S(j\omega)| d\omega \right| \leq \frac{3\delta\omega_c}{2k}.$$

The above result shows that the area of the tail of the Bode sensitivity integral over the infinite frequency range $[\omega_c, \infty)$ is *limited*.

Implication: Peak in the sensitivity frequency response before ω_c.

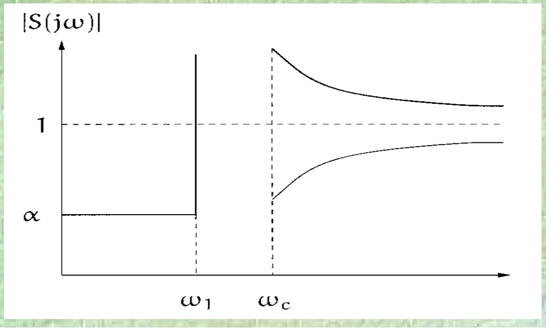
Suppose S satisfies the reduction spec.

$$|S(j\omega)| \le \alpha < 1, \qquad \forall \omega \le \omega_1 < \omega_c,$$

and translating

$$|L(j\omega)| \le \delta \left(\frac{\omega_c}{\omega} \right)^{1+k}, \qquad \forall \omega \ge \omega_c$$

on the shape of $|S(jw)|$ yields



Now, using the bounds (2) and (1) in the Bode sensitivity integral, it is easy to show that

$$\sup_{\omega \in (\omega_1, \omega_c)} \log |S(j\omega)| \geq \frac{1}{\omega_c - \omega_1} \left[\pi \sum_{p \in z_s} p + \omega_1 \log \frac{1}{\alpha} - \frac{3\delta\omega_c}{2k} \right].$$

Then, the larger the area of sensitivity reduction (i.e., α small and/or ω_1 close to ω_c) will necessarily result in a large peak in the range(ω_1, ω_c).

Hence, the Bode sensitivity integral imposes a design trade-off when natural bandwidth constraints are assumed for the closed-loop system.

Example

The inequality

$$\sup_{\omega \in (\omega_1, \omega_c)} \log |S(j\omega)| \geq \frac{1}{\omega_c - \omega_1} \left[\pi \sum_{p \in z_S} p + \omega_1 \log \frac{1}{\alpha} - \frac{3\delta\omega_c}{2k} \right].$$

can be used to derive a lower bound on the closed-loop bandwidth in terms of the sum of open-loop unstable poles. Let

$$\omega_1 = k_1 \omega_c.$$

Imposing the condition that the RHS above be yields the following lower bound on the bandwidth, which we take as

$$\omega_b \geq B(S_m) \sum_{p \in z_S} p,$$

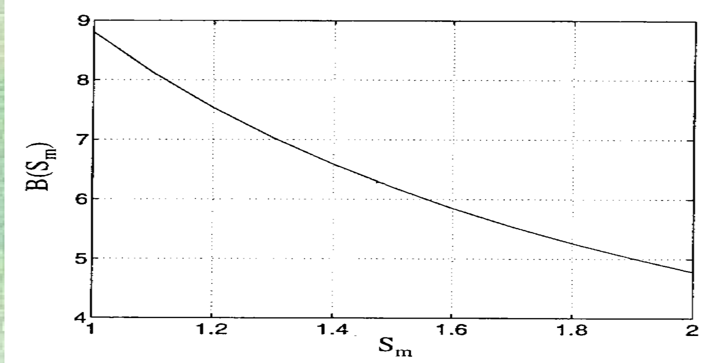
where

$$B(S_m) \stackrel{\Delta}{=} \frac{\pi}{(1-k_1)(S_m + 1.5\delta / k + k_1 \log \alpha)}.$$

$$\omega_b \geq B(S_m) \sum_{p \in Z_S} p$$

The factor $B(S_m)$ is plotted in the figure below as a function of the desired peak sensitivity S_m, for

$$\delta = 0.45,\ k = 1,\ k_1 = 0.7 \text{ and } \alpha = 0.5.$$



For example, an open-loop unstable system having relative degree two requires a bandwidth of at least 6.5 times the sum of its ORHP poles if it is desired that $|S|$ be smaller than 1/2 over 70% of the closed-loop bandwidth while keeping the lower bound on the peak sensitivity smaller than $S_m = \sqrt{2}$.

We next turn to dual results that hold for complementary sensitivity reduction.

Integral Constraints on Complementary Sensitivity

Consider a one d.o.f. stable control loop with open loop transfer function

$$G_o(s)C(s) = e^{-s\tau}H_{ol}(s) \qquad \tau \geq 0$$

where $H_{0l}(s)$ is a rational transfer function of relative degree $n_r > 1$ satisfying

$$H_{ol}(0)^{-1} = 0$$

Furthermore, assume that $H_{0l}(s)$ has no open loop zeros in the open RHP.

Then the nominal complementary sensitivity function satisfies:

$$\int_{0^-}^{\infty} \frac{1}{\omega^2} \ln |T_o(j\omega)| d\omega = \pi\tau - \frac{\pi}{2k_v}$$

where k_v is the velocity constant of the open loop transfer function satisfying:

$$\frac{1}{k_v} = -\lim_{s \to 0} \frac{dT(s)}{ds}$$

$$= -\lim_{s \to 0} \frac{1}{sH_{0l}(s)}$$

As for sensitivity reduction, the trade-off described above becomes harder in the presence of open loop RHP singularities. For the case of complementary sensitivity reduction, it is the open loop RHP zeros that influence the result. The formal result is stated on the next slide.

Assume that $H_{0l}(s)$ has open loop zeros in the open RHP, located at $c_1, c_2, \ldots, c_M$, then

$$\int_0^\infty \frac{1}{w^2} \ln |T_0(jw)| \, dw = \pi\tau + \pi \sum_{i=1}^{M} \frac{1}{c_i} - \frac{\pi}{2k_v}$$

We next turn to a set of frequency domain integrals which are closely related to the Bode type integrals presented above but which allow us to consider RHP open loop poles and zeros simultaneously. These integrals are usually called *Poisson type integral constraints*.

Poisson Integral Constraint on Sensitivity

A trick we will use here is to express a complex function *f*(*s*) as the product of functions which are non-minimum phase and analytic in the RHP, times the following Blaschke products (*or times the inverses of these products*).

$$B_z(s) = \prod_{k=1}^{M} \frac{s - c_k}{s + c_k^*} \qquad B_p(s) = \prod_{i=1}^{N} \frac{s - p_i}{s + p_i^*}$$

We use the above idea to prove the results presented below.

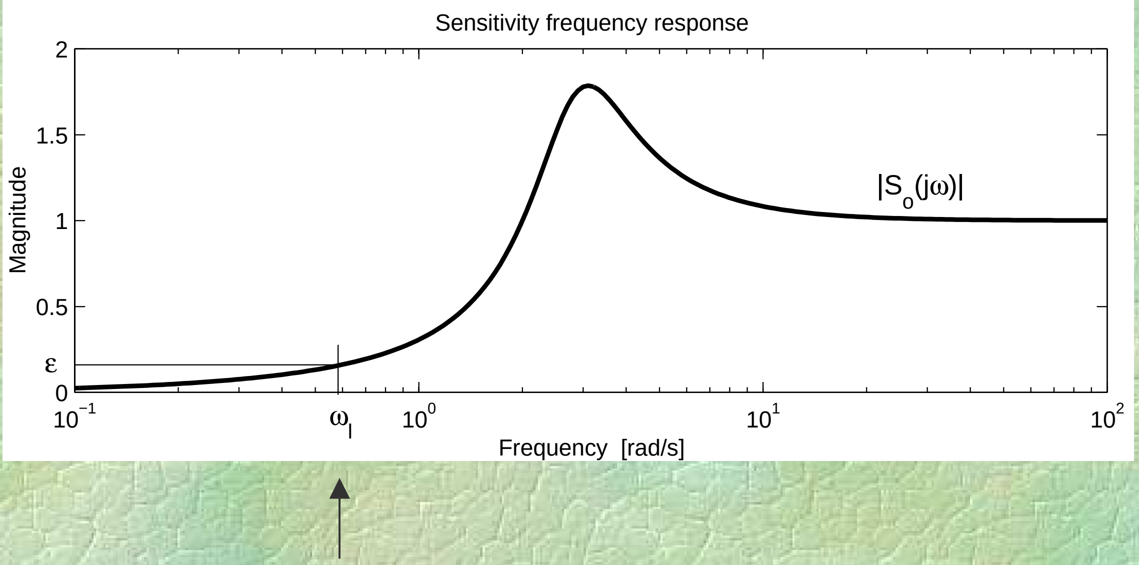
Poisson Integral for $S_0(jw)$

Consider a feedback control loop with open loop RHP zeros located at $c_1, c_2, \ldots, c_M$, where $c_k = \gamma_k + j\delta_k$ and open loop unstable poles located at $p_1, p_2, .., p_N$. Then the nominal sensitivity satisfies

$$\int_{-\infty}^{\infty} \ln|S_o(j\omega)| \frac{\gamma_k}{\gamma_k^2 + (\delta_k - \omega)^2} d\omega = -\pi \ln|B_p(c_k)| \qquad \text{for} \quad k = 1, 2, \ldots M$$

To illustrate this formula, consider the requirement that the sensitivity be reduced to below ε for all frequencies up to w_l. (*See the next slide*).

Figure 9.2: *Design specification for $|S_0(jw)|$*



Notional closed loop bandwidth

In the sequel we will need to use the integral of the term $W(c_k, \omega) \triangleq \dfrac{\gamma_k}{\gamma_k^2 + (\delta_k - \omega)^2}$

$$\int_{-\infty}^{\infty} W(c_k, \omega) d\omega = \pi$$

and we define

$$2\int_{\omega_1}^{\omega_2} W(c_k, \omega) d\omega = \Omega(c_k, \omega_2) - \Omega(c_k, \omega_1)$$

and

$$\Omega(c_k, \omega_c) = 2\arctan\left(\frac{\omega_c}{\gamma_k}\right) \qquad (*)$$

$$\Omega(c_k, \infty) = 2\lim_{\omega_c \to \infty} \arctan\left(\frac{\omega_c}{\gamma_k}\right) = \pi$$

Discussion

(i) Consider the plot of sensitivity versus frequency shown on the previous slide. Say we were to require the closed loop bandwidth to be greater than the magnitude of a right half plane (*real*) zero. In terms of the notation used in the figure, this would imply $w_l > \gamma_k$. We can then show using the Poisson formula that there is necessarily a very large sensitivity peak occurring beyond w_l. To estimate this peak, assume that $w_l = 2\gamma_k$ and take ϵ in Figure 6.2 as 0.3. Then, without considering the effect of any possible open loop unstable pole, the sensitivity peak will be bounded below as follows:

$$\ln S_{max} > \frac{1}{\pi - \Omega(c_k, 2c_k)} \left[|(\ln 0.3) \Omega(c_k, 2c_k)| \right]$$

Then, using equation (*) we have that $\Omega(c_k, 2c_k) = 2 \arctan(2) = 2.21$, leading to $S_{max} > 17.7$. That is we have a VERY large sensitivity peak. Note that this in turn implies that the complementary sensitivity peak will be bounded below by $S_{max} - 1 = 16.7$.

(ii) The observation in (i) is consistent with the analysis carried out in Chapter 8. In both cases the conclusion is that the closed loop bandwidth should not exceed the magnitude of the smallest RHP open loop zero. The penalty for not following this guideline is that a very large sensitivity peak will occur, leading to fragile loops (*non robust*) and large undershoots and overshoots.

(iii) In the presence of unstable open loop poles, the problem is compounded through the presence of the factor $|\ln|B_p(c_k)||$. This factor grows without bound when one RHP zero approaches an unstable open loop pole.

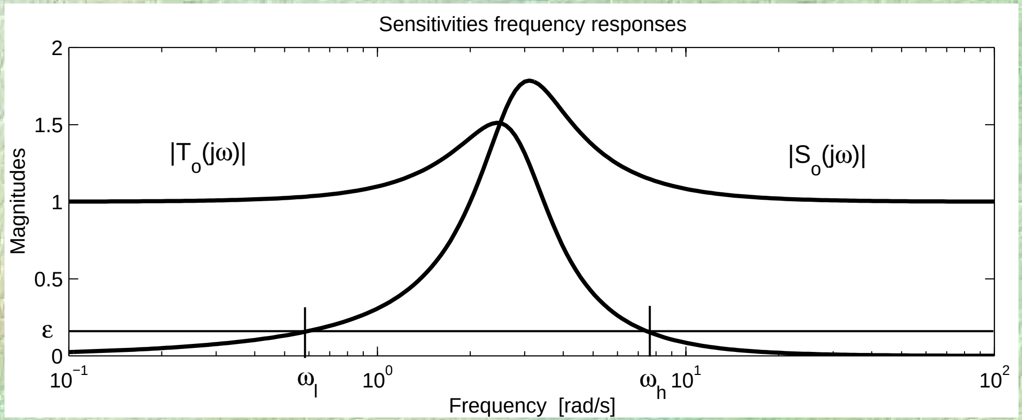
Constraints on both $|S_0|$ and $|T_0|$

Actually constraints can be imposed on both $|S_0|$ and $|T_0|$. For example, say that we require

$$|S_0(jw)| < \varepsilon \quad \text{for} \quad w < w_l$$
$$|T_0(jw)| < \varepsilon \quad \text{for} \quad w > w_h$$

This is illustrated on the next slide.

Figure 9.3: *Design specifications for $|S_0(jw)|$ and $|T_0(jw)|$*



We can then use the Poisson sensitivity integral to bound the peak sensitivity. The result is:

$$\ln S_{max} > \frac{1}{\Omega(c_k,\omega_h) - \Omega(c_k,\omega_l)} \left[\left| \pi \ln |B_p(c_k)| \right| + \left| (\ln \epsilon)\Omega(c_k,\omega_l) \right| \right.$$
$$\left. - \; (\pi - \Omega(c_k,\omega_h)) \ln(1+\epsilon) \right]$$

We next turn to the dual Poisson integral constraints that hold for the complementary sensitivity function. The formal result is stated on the next slide.

Poisson Integral Constraint on Complementary Sensitivity

Poisson integral for $T_0(jw)$. Consider a feedback control loop with delay $\tau \geq 0$, and having open loop unstable poles located at $p_1, p_2, \ldots, p_N$, where $p_i = \alpha_i + j\beta_i$ and open loop zeros in the open RHP, located at $c_1, c_2, \ldots, c_M$. Then,

$$\int_{-\infty}^{\infty} \ln|T_o(j\omega)| \frac{\alpha_i}{\alpha_i^2 + (\beta_i - \omega)^2} d\omega = -\pi \ln|B_z(p_i)| + \tau\alpha_i \qquad \text{for} \quad i = 1, 2, \ldots N$$

Say that we require that $|T_0| < \varepsilon$ for $w > w_h$. Then it follows from the above result that the peak value of the complementary sensitivity will be bounded from below as follows:

$$\ln T_{max} > \frac{1}{\Omega(\alpha_i, \omega_h)} [\pi |\ln |B_z(\alpha_i)|| + \tau\alpha_i + |\ln \epsilon|(\pi - \Omega(\alpha_i, \omega_h))]$$

Discussion

(i) We see that the lower bound on the complementary sensitivity peak is larger for systems with pure delays, and the influence of a delay increases for unstable poles which are far away from the imaginary axis, i.e. large α_i.

(ii) The peak, T_{max}, grows unbounded when a RHP zero approaches an unstable pole, since then $|\ln|B_z(p_i)||$ grows unbounded.

(iii) Say that we ask that the closed loop bandwidth be much smaller than the magnitude of a right half plane (*real*) pole. In terms of the notation used above, we would then have $w_h << \alpha_i$. Under these conditions, $\Omega(p_i, w_h)$ will be very small, leading to a very large complementary sensitivity peak. This is an unacceptable result. Thus we conclude that the closed loop bandwidth should be greater than any RHP open loop poles.

We note that this is consistent with the results based on time domain analysis presented in Chapter 8. There it was shown that, when the closed loop bandwidth is not greater than an open loop RHP pole, large time domain overshoot will occur.

Example of Design Trade-offs

We will illustrate the application of the above ideas by considering the inverted pendulum problem.

Such systems exist in many universities and are used to illustrate control principles. The key idea is to balance a rod on top of a moving cart (*similar to balancing a broom on one's hand*). A photograph of a real inverted pendulum system (*at the University of Newcastle*) is shown on the next slide.

Example of an Inverted Pendulum

We will consider the following control problem:

Sensors:

We measure the position of the cart (but NOT the angle of the pendulum)

Actuators:

We can apply forces to the cart.

Goal:

We want to position the cart at some location plus have the pendulum balancing vertically.

Inverted pendulum without angle measurement

A schematic diagram of the system is shown below:

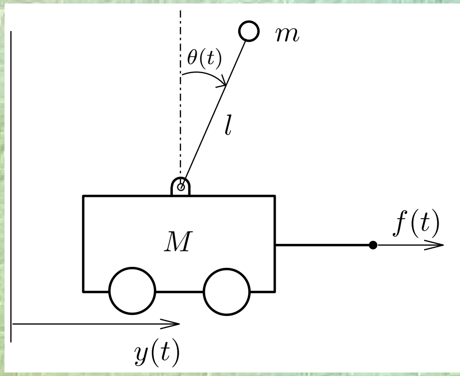


Figure 9.4: *Inverted pendulum*

The model for this system was discussed in Chapter 3. A linearized model for the system has the following transfer function linking the cart position (Y) to the force applied to the cart (F).

$$\frac{Y(s)}{F(s)} = 2\frac{(s-\sqrt{10})(s+\sqrt{10})}{s^2(s-\sqrt{20})(s+\sqrt{20})}$$

We note that this system has

- *an open loop RHP zero at* $\sqrt{10}$
- *an open loop RHP pole at* $\sqrt{20}$

We observe that the RHP pole has greater magnitude than the RHP zero. The reader is reminded of the comments made in the following two slides which appeared in the earlier part of this chapter.

Indeed, it will turn out that to avoid large frequency domain sensitivity peaks it is necessary to limit the range of sensitivity reduction to be:

(i) *less than any right half plane open loop zero*

(ii) *greater than any right half plane open loop pole.*

This begs the question - "*What happens if there is a right half plane open loop zero having smaller magnitude than a right half plan open loop pole?*"

Clearly the requirements specified on the previous slide are then mutually incompatible. The consequence is that large sensitivity peaks are unavoidable and, as a result, poor feedback performance is inevitable.

An example precisely illustrating this conclusion will be presented later.

We will show formally using the Poisson integral formulae that the predictions made above are indeed true for this example.

We consider various choices for w_l and w_h with $\epsilon = 0.1$

- $w_l = \sqrt{10}$ and $w_h = 100$. Then the equation for S_{max} predicts that $S_{max} \geq 432$. In this case, w_h is much larger than the unstable pole, thus the large value for the bound results from $w_l = \sqrt{10}$ being too close to the NMP zero.
- When $w_l = 1$ and $w_h = 100$, we have that $S_{max} \geq 16.7$, which is significantly lower than the previous case (*although still very large*), since now, w_l is much smaller than the NMP zero.
- If $w_l = 1$, and $w_h = \sqrt{20}$ we obtain that $T_{max} \geq 3171$, which is due to the fact that w_h is too close to the unstable pole.

- If $w_l = 1$, and $w_h = 3$ we obtain that $T_{max} \geq 7.2 \times 10^5$. This huge lower bound originates from two facts: firstly w_h is lower than the unstable pole, and secondly, w_l and w_h are very close.

We thus see that, no matter how we try to allocate w_l and w_h, large sensitivity peaks occur. Thus this system seems to be extremely difficult to control.

The above conclusion is not unreasonable. (The reader should try balancing a broom on one's hand with one's eyes shut!)

The key point is that the angle of the pendulum is not measured.

In a later chapter we will see that a simple change in the architecture made possible by providing a measurement of the angle turns this near impossible control problem into a very easy one (*see also the discussion on the web site*).

To illustrate just how hard this problem is, (*when the angle is not measured*) we designed a stabilizing controller as shown on the next slide.

$$C(s) = \frac{p_3 s^3 + p_2 s^2 + p_1 s + p_o}{s^s + \ell_2 s^2 + \ell_1 s + \ell_o}$$

Where

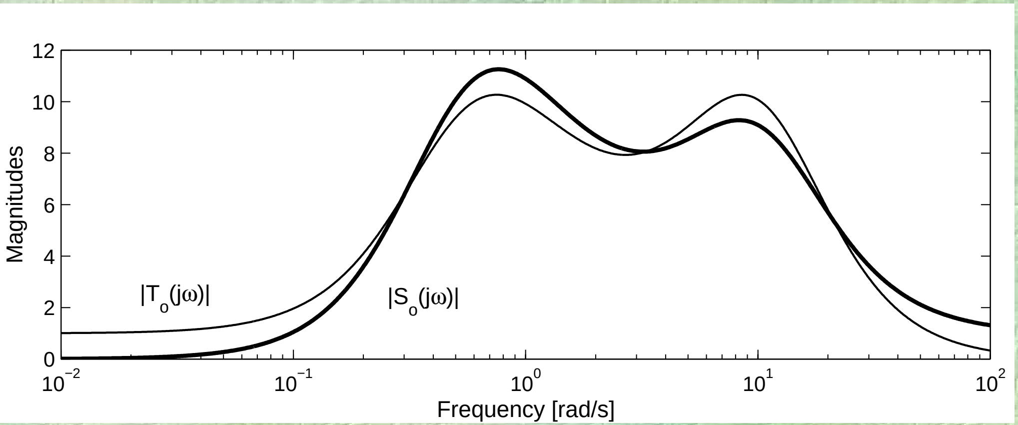
$$\begin{aligned}
p_0 &= -74.3 \\
p_1 &= -472.8 \\
p_2 &= 7213.0 \\
p_3 &= 1690.5 \\
\ell_0 &= -8278.9 \\
\ell_1 &= -2682.3 \\
\ell_2 &= 41.5
\end{aligned}$$

Using the Poisson integrals, we predict

$$S_{max} \geq 6.34$$
$$T_{max} \geq 7.19$$

The actual sensitivity plots are shown on the next page. These show that these lower bounds are indeed exceeded by the specified controller presented on the previous slide.

Figure 9.5: *Sensitivities for the inverted pendulum*



Summary

- One class of design constraints are those which hold at a particular frequency.
- Thus we can view the law $S(j_w) = 1 - T(j_w)$ on a frequency by frequency basis. It states that no single frequency can be removed from both the sensitivity, $S(j_w)$, and complementary sensitivity, $T(j_w)$.
- There is, however, an additional class of design considerations, which results from so called frequency domain integral constraints, see Table 9.1.

Table 9.1

Notation	*Constraints*	*Interpretation*
p_i RHP poles	$$\int_0^\infty ln\|S_o(j\omega)\|d\omega = \pi\sum_{i=1}^{N}\Re\{p_i\}$$	Areas of sensitivity below 1 (i.e. $ln\|S_o\| < 0$) must be matched by areas of sensitivity above 1.
c_i RHP zeros	$$\int_0^\infty \frac{ln\|T_o(j\omega)\|}{\omega^2}d\omega = \pi\sum_{i=0}^{N}\frac{1}{c_i} - \frac{\pi}{2k_v}$$	Areas of complementary sensitivity below 1 must be matched by areas of complementary sensitivity above 1.
$W(c_k,\omega)$ weighting function $B_p(c_k)$ Blaschke product	$$2\int_0^\infty ln\|S_o(j\omega)\|W(c_k,\omega)d\omega = -\pi ln\|B_p(c_k)\|$$	The above sensitivity trade-off is focused on areas near RHP zeros.
$W(p_i,\omega)$ weighting function $B_z(p_i)$ Blaschke product	$$2\int_0^\infty ln\|T_o(j\omega)\|W(p_i,\omega)d\omega = -\pi ln\|B_z(p_i)\|$$	The above complementary sensitivity trade-off is focused on areas near RHP poles.

- This chapter explores the origin and nature of these integral constraints and derives their implications for control system performance:
  - The constraints are a direct consequence of the requirement that all sensitivity functions must be stable;
  - mathematically, this means that the sensitivities are required to be analytic in the right half complex plane;
  - results from analytic function theory then show, that this requirement necessarily implies weighted integrals of the frequency response necessarily evaluate to a constant;
  - hence, if one designs a controller to have low sensitivity in a particular frequency range, then the sensitivity will necessarily increase at other frequencies-a consequence of the weighted integral always being a constant; this phenomenon has also been called the water bed effect (pushing down on the water bed in one area, raises it somewhere else).

- These trade-offs show that systems become increasingly difficult to control as
  - Unstable zeros become slower
  - Unstable poles become faster
  - Time delays get bigger.