Appendix D

PROPERTIES OF CONTINUOUS-TIME RICCATI EQUATIONS

This appendix summarizes key properties of the Continuous-Time Differential Riccati Equation (CTDRE);

$$\frac{d\mathbf{P}}{dt} = -\mathbf{A}^T\mathbf{P}(t) - \mathbf{P}(t)\mathbf{A} + \mathbf{P}(t)\mathbf{B}\boldsymbol{\Phi}^{-1}\mathbf{B}^T\mathbf{P}(t) - \boldsymbol{\Psi} \tag{D.0.1}$$

$$\mathbf{P}(t_f) = \boldsymbol{\Psi}_f \tag{D.0.2}$$

and the Continuous-Time Algebraic Riccati Equation (CTARE)

$$0 = -\mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{A} + \mathbf{P}\mathbf{B}\boldsymbol{\Phi}^{-1}\mathbf{B}^T\mathbf{P} - \boldsymbol{\Psi} \tag{D.0.3}$$

D.1 Solutions of the CTDRE

The following lemma gives a useful alternative expression for $\mathbf{P}(t)$.

Lemma D.1. *The solution,* $\mathbf{P}(t)$*, to the CTDRE* (D.0.1)*, can be expressed as*

$$\mathbf{P}(t) = \mathbf{N}(t)[\mathbf{M}(t)]^{-1} \tag{D.1.1}$$

where $\mathbf{M}(t) \in \mathbb{R}^{n\times n}$ *and* $\mathbf{N}(t) \in \mathbb{R}^{n\times n}$ *satisfy the following equation:*

$$\frac{d}{dt}\begin{bmatrix}\mathbf{M}(t)\\ \mathbf{N}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A} & -\mathbf{B}\boldsymbol{\Phi}^{-1}\mathbf{B}^T\\ -\boldsymbol{\Psi} & -\mathbf{A}^T\end{bmatrix}\begin{bmatrix}\mathbf{M}(t)\\ \mathbf{N}(t)\end{bmatrix} \tag{D.1.2}$$

subject to

$$\mathbf{N}(t_f)[\mathbf{M}(t_f)]^{-1} = \boldsymbol{\Psi}_f \tag{D.1.3}$$

Proof

We show that $\mathbf{P}(t)$*, as defined above, satisfies the CTDRE. We first have that*

$$\frac{d\mathbf{P}(t)}{dt} = \frac{d\mathbf{N}(t)}{dt}[\mathbf{M}(t)]^{-1} + \mathbf{N}(t)\frac{d[\mathbf{M}(t)]^{-1}}{dt} \tag{D.1.4}$$

The derivative of $[\mathbf{M}(t)]^{-1}$ *can be computed by noting that* $\mathbf{M}(t)[\mathbf{M}(t)]^{-1} = \mathbf{I}$*; then*

$$\frac{d\mathbf{I}}{dt} = \mathbf{0} = \frac{d\mathbf{M}(t)}{dt}[\mathbf{M}(t)]^{-1} + \mathbf{M}(t)\frac{d[\mathbf{M}(t)]^{-1}}{dt} \tag{D.1.5}$$

from which we obtain

$$\frac{d[\mathbf{M}(t)]^{-1}}{dt} = -[\mathbf{M}(t)]^{-1}\frac{d\mathbf{M}(t)}{dt}[\mathbf{M}(t)]^{-1} \tag{D.1.6}$$

Thus, equation (D.1.4) *can be used with* (D.1.2) *to yield*

$$\begin{aligned} -\frac{d\mathbf{P}(t)}{dt} &= \mathbf{A}^T\mathbf{N}(t)[\mathbf{M}(t)]^{-1} + \mathbf{N}(t)[\mathbf{M}(t)]^{-1}\mathbf{A} + \mathbf{\Psi} \\ &\quad - \mathbf{N}(t)[\mathbf{M}(t)]^{-1}\mathbf{B}[\Phi]^{-1}\mathbf{B}^T\mathbf{N}(t)[\mathbf{M}(t)]^{-1} \end{aligned} \tag{D.1.7}$$

which shows that $\mathbf{P}(t)$ *also satisfies* (D.0.1)*, upon using* (D.1.1).

The matrix on the right-hand side of (D.1.2), namely,

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{\Phi}^{-1}\mathbf{B}^T \\ -\mathbf{\Psi} & -\mathbf{A}^T \end{bmatrix} \qquad \mathbf{H} \in \mathbb{R}^{2n\times 2n} \tag{D.1.8}$$

is called the *Hamiltonian* matrix associated with this problem.

Next, note that (D.0.1) can be expressed in compact form as

$$\begin{bmatrix} -\mathbf{P}(t) & \mathbf{I} \end{bmatrix} \mathbf{H} \begin{bmatrix} \mathbf{I} \\ \mathbf{P}(t) \end{bmatrix} = \frac{d\mathbf{P}(t)}{dt} \tag{D.1.9}$$

Then, not surprisingly, solutions to the CTDRE, (D.0.1), are intimately connected to the properties of the Hamiltonian matrix.

We first note that $\mathbf{H}$ has the following reflexive property:

$$\mathbf{H} = -\mathbf{T}\mathbf{H}^T\mathbf{T}^{-1} \qquad \text{with} \qquad \mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{I_n} \\ -\mathbf{I_n} & \mathbf{0} \end{bmatrix} \tag{D.1.10}$$

where $\mathbf{I_n}$ is the identity matrix in $\mathbb{R}^{n\times n}$.

Recall that a similarity transformation preserves the eigenvalues; thus, the eigenvalues of $\mathbf{H}$ are the same as those of $-\mathbf{H}^T$. On the other hand, the eigenvalues of $\mathbf{H}$ and $\mathbf{H}^T$ must be the same. Hence, the spectral set of $\mathbf{H}$ is the union of two sets, Λ_s and Λ_u, such that, if $\lambda \in \Lambda_s$, then $-\lambda \in \Lambda_u$. We assume that $\mathbf{H}$ does not contain any eigenvalue on the imaginary axis (note that it suffices, for this to occur, that $(\mathbf{A}, \mathbf{B})$ be stabilizable and that the pair $(\mathbf{A}, \mathbf{\Psi}^{\frac{1}{2}})$ have no undetectable poles on the stability boundary). In this case, Λ_s can be so formed that it contains only the eigenvalues of $\mathbf{H}$ that lie in the open LHP. Then, there always exists a nonsingular transformation $\mathbf{V} \in \mathbb{R}^{2n\times 2n}$ such that

$$[\mathbf{V}]^{-1}\mathbf{H}\mathbf{V} = \begin{bmatrix} \mathbf{H_s} & \mathbf{0} \\ \mathbf{0} & \mathbf{H_u} \end{bmatrix} \tag{D.1.11}$$

where $\mathbf{H_s}$ and $\mathbf{H_u}$ are diagonal matrices with eigenvalue sets Λ_s and Λ_u, respectively.

We can use $\mathbf{V}$ to transform the matrices $\mathbf{M}(t)$ and $\mathbf{N}(t)$, to obtain

$$\begin{bmatrix} \tilde{\mathbf{M}}(t) \\ \tilde{\mathbf{N}}(t) \end{bmatrix} = [\mathbf{V}]^{-1} \begin{bmatrix} \mathbf{M}(t) \\ \mathbf{N}(t) \end{bmatrix} \tag{D.1.12}$$

Thus, (D.1.2) can be expressed in the equivalent form:

$$\frac{d}{dt}\begin{bmatrix} \tilde{\mathbf{M}}(t) \\ \tilde{\mathbf{N}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{H_s} & \mathbf{0} \\ \mathbf{0} & \mathbf{H_u} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{M}}(t) \\ \tilde{\mathbf{N}}(t) \end{bmatrix} \tag{D.1.13}$$

If we partition $\mathbf{V}$ in a form consistent with the matrix equation (D.1.13), we have that

$$\mathbf{V} = \begin{bmatrix} \mathbf{V_{11}} & \mathbf{V_{12}} \\ \mathbf{V_{21}} & \mathbf{V_{22}} \end{bmatrix} \tag{D.1.14}$$

The solution to the CTDRE is then given by the following lemma.

Lemma D.2. *A solution for equation* (D.0.1) *is given by*

$$\mathbf{P}(t) = \mathbf{P_1}(t)[\mathbf{P_2}(t)]^{-1} \tag{D.1.15}$$

where

$$\mathbf{P_1}(t) = \mathbf{V_{21}} + \mathbf{V_{22}}e^{-\mathbf{H_u}(t_f-t)}\mathbf{V_a}e^{\mathbf{H_s}(t_f-t)} \tag{D.1.16}$$

$$\mathbf{P_2}(t) = \left[\mathbf{V_{11}} + \mathbf{V_{12}}e^{-\mathbf{H_u}(t_f-t)}\mathbf{V_a}e^{\mathbf{H_s}(t_f-t)}\right]^{-1} \tag{D.1.17}$$

$$\mathbf{V_a} \stackrel{\triangle}{=} -[\mathbf{V_{22}} - \mathbf{\Psi}_f\mathbf{V_{12}}]^{-1}[\mathbf{V_{21}} - \mathbf{\Psi}_f\mathbf{V_{11}}] = \tilde{\mathbf{N}}(t_f)\left[\tilde{\mathbf{M}}(t_f)\right]^{-1} \tag{D.1.18}$$

Proof

From (D.1.12), we have

$$\begin{aligned}\mathbf{M}(t_f) &= \mathbf{V}_{11}\tilde{\mathbf{M}}(t_f) + \mathbf{V}_{12}\tilde{\mathbf{N}}(t_f) \\ \mathbf{N}(t_f) &= \mathbf{V}_{21}\tilde{\mathbf{M}}(t_f) + \mathbf{V}_{22}\tilde{\mathbf{N}}(t_f)\end{aligned} \tag{D.1.19}$$

Hence, from (D.1.3),

$$\left[\mathbf{V}_{21}\tilde{\mathbf{M}}(t_f) + \mathbf{V}_{22}\tilde{\mathbf{N}}(t_f)\right]\left[\mathbf{V}_{11}\tilde{\mathbf{M}}(t_f) + \mathbf{V}_{12}\tilde{\mathbf{N}}(t_f)\right]^{-1} = \mathbf{\Psi}_f \tag{D.1.20}$$

or

$$\left[\mathbf{V}_{21} + \mathbf{V}_{22}\tilde{\mathbf{N}}(t_f)[\tilde{\mathbf{M}}(t_f)]^{-1}\right]\left[\mathbf{V}_{11} + \mathbf{V}_{12}\tilde{\mathbf{N}}(t_f)[\tilde{\mathbf{M}}(t_f)]^{-1}\right]^{-1} = \mathbf{\Psi}_f \tag{D.1.21}$$

or

$$\tilde{\mathbf{N}}(t_f)[\tilde{\mathbf{M}}(t_f)]^{-1} = -\left[\mathbf{V}_{22}\Psi_f\mathbf{V}_{12}\right]^{-1}\left[\mathbf{V}_{21} - \Psi_f\mathbf{V}_{11}\right] = \mathbf{V_a} \tag{D.1.22}$$

Now, from (D.1.10),

$$\begin{aligned}\mathbf{P}(t) &= \mathbf{N}(t)[\mathbf{M}(t)]^{-1} \\ &= \left[\mathbf{V}_{21}\tilde{\mathbf{M}}(t) + \mathbf{V}_{22}\tilde{\mathbf{N}}(t)\right]\left[\mathbf{V}_{11}\tilde{\mathbf{M}}(t) + \mathbf{V}_{12}\tilde{\mathbf{N}}(t)\right]^{-1} \\ &= \left[\mathbf{V}_{21} + \mathbf{V}_{22}\tilde{\mathbf{N}}(t)[\tilde{\mathbf{M}}(t)]^{-1}\right]\left[\mathbf{V}_{11} + \mathbf{V}_{12}\tilde{\mathbf{N}}(t)[\tilde{\mathbf{M}}(t)]^{-1}\right]^{-1}\end{aligned} \tag{D.1.23}$$

and the solution to (D.1.13) is

$$\begin{aligned}\tilde{\mathbf{M}}(t_f) &= e^{\mathbf{H}_s(t_f-t)}\tilde{\mathbf{M}}(t) \\ \tilde{\mathbf{N}}(t_f) &= e^{\mathbf{H}_u(t_f-t)}\tilde{\mathbf{N}}(t)\end{aligned} \tag{D.1.24}$$

Hence,

$$\tilde{\mathbf{N}}(t)[\tilde{\mathbf{M}}(t)]^{-1} = e^{-\mathbf{H}_u(t_f-t)}\tilde{\mathbf{N}}(t_f)[\tilde{\mathbf{M}}(t_f)]^{-1}e^{\mathbf{H}_s(t_f-t)} \tag{D.1.25}$$

Substituting (D.1.25) into (D.1.23) gives the result.

□□□

D.2 Solutions of the CTARE

The **C**ontinuous **T**ime **A**lgebraic **R**iccati **E**quation (CTARE) has many solutions, because it is a matrix quadratic equation. The solutions can be characterized as follows.

Lemma D.3. *Consider the following CTARE:*

$$0 = \mathbf{\Psi} - \mathbf{P}\mathbf{B}\mathbf{\Phi}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} \tag{D.2.1}$$

(i) The CTARE can be expressed as

$$\begin{bmatrix} -\mathbf{P} & \mathbf{I} \end{bmatrix} \mathbf{H} \begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix} = \mathbf{0} \tag{D.2.2}$$

where $\mathbf{H}$ *is defined in (D.1.8).*

(ii) Let $\overline{\mathbf{V}}$ *be defined so that*

$$\overline{\mathbf{V}}^{-1}\mathbf{H}\overline{\mathbf{V}} = \begin{bmatrix} \mathbf{\Lambda}_a & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_b \end{bmatrix} \tag{D.2.3}$$

where $\mathbf{\Lambda}_a, \mathbf{\Lambda}_b$ *are any partitioning of the (generalized) eigenvalues of* $\mathbf{H}$ *such that, if* λ *is equal to* $(\Lambda_a)_i$ *for same* i*, then* $-\lambda^* = (\Lambda_b)_j$ *for some* j*.*

Let

$$\overline{\mathbf{V}} = \begin{bmatrix} \overline{\mathbf{V}}_{11} & \overline{\mathbf{V}}_{12} \\ \overline{\mathbf{V}}_{21} & \overline{\mathbf{V}}_{22} \end{bmatrix} \tag{D.2.4}$$

Then $\overline{\mathbf{P}} = \overline{\mathbf{V}}_{21}\overline{\mathbf{V}}_{11}^{-1}$ *is a solution of the CTARE.*

Proof

(i) This follows direct substitution.

(ii) The form of $\overline{\mathbf{P}}$ *ensures that*

$$\begin{bmatrix} -\overline{\mathbf{P}} & \mathbf{I} \end{bmatrix} \overline{\mathbf{V}} = \begin{bmatrix} \mathbf{0} & * \end{bmatrix} \tag{D.2.5}$$

$$\overline{\mathbf{V}}^{-1} \begin{bmatrix} \overline{\mathbf{P}} \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} * \\ \mathbf{0} \end{bmatrix} \tag{D.2.6}$$

*where * denotes a possible nonzero component.*
Hence,

$$\begin{bmatrix} -\overline{\mathbf{P}} & \mathbf{I} \end{bmatrix} \overline{\mathbf{V}} \mathbf{\Lambda} \mathbf{V}^{-1} \begin{bmatrix} \mathbf{P} \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & * \end{bmatrix} \mathbf{\Lambda} \begin{bmatrix} * \\ \mathbf{0} \end{bmatrix} \tag{D.2.7}$$

$$= 0 \tag{D.2.8}$$

□□□

D.3 The stabilizing solution of the CTARE

We see from Section D.2 that we have as many solutions to the CTARE as there are ways of partitioning the eigevalues of H into the groups $\mathbf{\Lambda}_a$ and $\mathbf{\Lambda}_b$. Provided that (A, B) is stabilizable and that $(\mathbf{\Psi}^{\frac{1}{2}}, \mathbf{A})$ has no unobservable modes in the imaginary axis, then $\mathbf{H}$ has no eigenvalues in the imaginary axis. In this case, there exists a unique way of partitioning the eigenvalues so that $\mathbf{\Lambda}_a$ contains only the stable eigenvalues of $\mathbf{H}$. We call the corresponding (unique) solution of the CTARE *the stabilizing solution* and denote it by $\mathbf{P}_\infty^s$.
Properties of the stabilizing solution are given in the following.

Lemma D.4. *(a) The stabilizing solution has the property that the closed loop* $\mathbf{A}$ *matrix,*

$$\mathbf{A_{cl}} = \mathbf{A} - \mathbf{B}\mathbf{K}_\infty^s \tag{D.3.1}$$

where

$$K_\infty^s = \Phi^{-1} B^T \mathbf{P}_\infty^s \tag{D.3.2}$$

has eigenvalues in the open left-half plane.

(b) If $(\mathbf{\Psi}^{\frac{1}{2}}, \mathbf{A})$ *is detectable, then the stabilizing solution is the only nonnegative solution of the CTARE.*

(c) If $(\mathbf{\Psi}^{\frac{1}{2}}, \mathbf{A})$ *has no unobservable modes inside the stability boundary, then the stabilizing solution is positive definite, and conversely.*

(d) If $(\Psi^{\frac{1}{2}}, A)$ *has an unobservable mode outside the stabilizing region, then in addition to the stabilizing solution, there exists at least one other nonnegative solution of the CTARE. However, the stabilizing solution,* $\mathbf{P}_\infty^s$ *has the property that*

$$\mathbf{P}_\infty^s - \mathbf{P}_\infty' \geq \mathbf{0} \tag{D.3.3}$$

where $\mathbf{P}_\infty'$ *is any other solution of the CTARE.*

Proof

For part (a), we argue as follows:

Consider (D.1.11) and (D.1.14). Then

$$\mathbf{H}\begin{bmatrix}\mathbf{V_{11}}\\ \mathbf{V_{21}}\end{bmatrix} = \begin{bmatrix}\mathbf{V_{11}}\\ \mathbf{V_{21}}\end{bmatrix}\mathbf{H_s} \tag{D.3.4}$$

which implies that

$$\mathbf{H}\begin{bmatrix}\mathbf{I}\\ \mathbf{V_{21}}\mathbf{V_{11}}^{-1}\end{bmatrix} = \mathbf{H}\begin{bmatrix}\mathbf{I}\\ \mathbf{P}_\infty\end{bmatrix} = \begin{bmatrix}\mathbf{V_{11}}\mathbf{H_s}\mathbf{V_{11}}^{-1}\\ \mathbf{V_{21}}\mathbf{H_s}\mathbf{V_{11}}^{-1}\end{bmatrix} \tag{D.3.5}$$

If we consider only the first row in (D.3.5), then, using (D.1.8), we have

$$\mathbf{V_{11}}\mathbf{H_s}\mathbf{V_{11}}^{-1} = \mathbf{A} - \mathbf{B}\mathbf{\Phi}^{-1}\mathbf{B}^T\mathbf{P}_\infty = \mathbf{A} - \mathbf{B}\mathbf{K} \tag{D.3.6}$$

Hence, the closed-loop poles are the eigenvalues of $\mathbf{H_s}$ *and, by construction, these are stable.*

We leave the reader to pursue parts (b), (c), and (d) by studying the references given at the end of Chapter 24.

□□□

D.4 Convergence of Solutions of the CTARE to the Stabilizing Solution of the CTARE

Finally, we show that, under reasonable conditions, the solution of the CTDRE will converge to the unique stabilizing solution of the CTARE. In the sequel, we will be particularly interested in the stabilizing solution to the CTARE.

Lemma D.5. *Provided that* (A, B) *is stabilizable and that* $(\mathbf{\Psi}^{\frac{1}{2}}, \mathbf{A})$ *has no unobservable poles on the imaginary axis and that* $\mathbf{\Psi}_f > \mathbf{P}_\infty^s$*, then*

$$\lim_{t_f\to\infty} \mathbf{P}(t) = \mathbf{P}_\infty^s \tag{D.4.1}$$

Proof

We observe that the eigenvalues of $\mathbf{H}$ *can be grouped so that* $\mathbf{\Lambda}_s$ *contains only eigenvalues that lie in the left-half plane. We then have that*

$$\lim_{t_f\to\infty} e^{\mathbf{H_s}(t_f-t)} = \mathbf{0} \qquad \text{and} \qquad \lim_{t_f\to\infty} e^{-\mathbf{H_u}(t_f-t)} = \mathbf{0} \tag{D.4.2}$$

given that $\mathbf{H}_s$ *and* $-\mathbf{H}_u$ *are matrices with eigenvalues strictly inside the LHP.*

The result then follows from (D.1.16) to (D.1.17).

Remark D.1. *Actually, provided that* $(\Psi^{\frac{1}{2}}, A)$ *is detectable, then it suffices to have* $\mathbf{\Psi}_f \geq \mathbf{0}$ *in Lemma D.5*

□□□

D.5 Duality between Linear Quadratic Regulator and Optimal Linear Filter

The close connections between the optimal filter and the LQR problem can be expressed directly as follows: We consider the problem of estimating a particular linear combination of the states, namely,

$$z(t) = f^T x(t) \tag{D.5.1}$$

(The final solution will turn out to be independent of f, and thus will hold for the complete state vector.)

Now we will estimate $z(t)$ by using a linear filter of the following form:

$$\hat{z}(t) = \int_0^t h(t-\tau)^T y'(\tau) d\tau + g^T \hat{x}_o \tag{D.5.2}$$

where $h(t)$ is the impulse response of the filter and where $\hat{x}_o$ is a given estimate of the initial state. Indeed, we will assume that (22.10.17) holds, that is, that the initial state $x(0)$ satisfies

$$\mathcal{E}(x(0) - \hat{x}_o)(x(0) - \hat{x}_o)^T = \mathbf{P_o} \tag{D.5.3}$$

We will be interested in designing the filter impulse response, $h(\tau)$, so that $\hat{z}(t)$ is *close* to $z(t)$ in some sense. (Indeed, the precise sense we will use is a quadratic form.) From (D.5.1) and (D.5.2), we see that

$$\begin{aligned} \tilde{z}(t) &= z(t) - \hat{z}(t) \\ &= f^T x(t) - \int_0^t h(t-\tau)^T y'(\tau) d\tau - g^T \hat{x}_o \\ &= f^T x(t) - \int_0^t h(t-\tau)^T \big(\mathbf{C}x(\tau) + \dot{v}(t)\big) d\tau - g^T \hat{x}_o \end{aligned} \tag{D.5.4}$$

Equation (D.5.4) is somewhat difficult to deal with, because of the cross-product between $h(t-\tau)$ and $x(t)$ in the integral. Hence, we introduce another variable, λ, by using the following equation

$$\frac{d\lambda(\tau)}{d\tau} = -\mathbf{A}^T \lambda(\tau) - \mathbf{C}^T u(\tau) \tag{D.5.5}$$

$$\lambda(t) = -f \tag{D.5.6}$$

where $u(\tau)$ is the reverse time form of h:

$$u(\tau) = h(t-\tau) \tag{D.5.7}$$

Substituting (D.5.5) into (D.5.4) gives

$$\begin{aligned}\tilde{z}(t) =& f^T x(t) + \int_0^t \left[\frac{d\lambda(\tau)}{d\tau} + \mathbf{A}^T\lambda(\tau)\right]^T x(\tau)d\tau \\ &- \int_0^t u(\tau)\dot{v}(\tau)d\tau - g^T\hat{x}_o\end{aligned} \tag{D.5.8}$$

Using integration by parts, we then obtain

$$\begin{aligned}\tilde{z}(t) =& f^T x(t) + \left[\lambda^T x(\tau)\right]_0^t - g^T\hat{x}_o \\ &+ \int_0^t \left(-\lambda(\tau)^T\frac{dx(\tau)}{d\tau} + \lambda(\tau)^T\mathbf{A}x(\tau) - u(\tau)^T\frac{dv(\tau)}{d\tau}\right)d\tau\end{aligned} \tag{D.5.9}$$

Finally, using (22.10.5) and (D.5.6), we obtain

$$\begin{aligned}\tilde{z}(t) =& \lambda(0)^T(x(0)-\hat{x}_o) + \int_0^t \left(-\lambda(\tau)^T\frac{dw(\tau)}{d\tau} - u(\tau)^T\frac{dv(\tau)}{d\tau}\right)d\tau \\ &- (\lambda(0)+g)^T\hat{x}_o\end{aligned} \tag{D.5.10}$$

Hence, squaring and taking mathematical expectations, we obtain (upon using (D.5.3), (22.10.3), and (22.10.4)) the following:

$$\begin{aligned}\mathcal{E}\{\tilde{z}(t)^2\} =& \lambda(0)^T\mathbf{P_o}\lambda(0) + \int_0^t \big(\lambda(\tau)^T\mathbf{Q}\lambda(\tau) + u(\tau)^T\mathbf{R}u(\tau)\big)d\tau \\ &+ \parallel (\lambda(0)+g)^T\hat{x}_o \parallel^2\end{aligned} \tag{D.5.11}$$

The last term in (D.5.11) is zero if $g = -\lambda(0)$. Thus, we see that the design of the optimal linear filter can be achieved by minimizing

$$J = \lambda(0)^T\mathbf{P_o}\lambda(0) + \int_0^t \big(\lambda(\tau)^T\mathbf{Q}\lambda(\tau) + u(\tau)^T\mathbf{R}u(\tau)\big)d\tau \tag{D.5.12}$$

where $\lambda(\tau)$ satisfies the reverse-time equations (D.5.5) and (D.5.6).

We recognize the set of equations formed by (D.5.5), (D.5.6), and (D.5.12) as a **standard linear regulator problem**, provided that the connections shown in Table D.1 are made.

Finally, by using the (dual) optimal control results presented earlier, we see that the optimal filter is given by

Regulator	Filter	Regulator	Filter
τ	$t-\tau$	t_f	0
$\mathbf{A}$	$-\mathbf{A}^T$	$\mathbf{\Psi}$	$\mathbf{Q}$
$\mathbf{B}$	$-\mathbf{C}^T$	$\mathbf{\Phi}$	$\mathbf{R}$
x	λ	$\mathbf{\Psi}_f$	$\mathbf{P_o}$

Table D.1. Duality in quadratic regulators and filters

$$\hat{z}^o(\tau) = \int_o^t u^o(\tau)^T y'(\tau)d\tau + g^T\hat{x}_o \tag{D.5.13}$$

where

$$u^o(\tau) = -\mathbf{K_f}(\tau)\lambda(\tau) \tag{D.5.14}$$
$$\mathbf{K_f}(\tau) = \mathbf{R}^{-1}\mathbf{C}\mathbf{\Sigma}(\tau) \tag{D.5.15}$$

and $\mathbf{\Sigma}(\tau)$ satisfies the dual form of (D.0.1), (22.4.18):

$$-\frac{d\mathbf{\Sigma}(t)}{dt} = \mathbf{Q} - \mathbf{\Sigma}(t)\mathbf{C}^T\mathbf{R}^{-1}\mathbf{C}\mathbf{\Sigma}(t) + \mathbf{\Sigma}(t)\mathbf{A}^T + \mathbf{A}\mathbf{\Sigma}(t) \tag{D.5.16}$$

$$\mathbf{\Sigma}(0) = \mathbf{P_o} \tag{D.5.17}$$

Substituting (D.5.14) into (D.5.5), (D.5.6) we see that

$$\frac{d\lambda(\tau)}{d\tau} = -\mathbf{A}^T\lambda(\tau) + \mathbf{C}^T\mathbf{K_f}(\tau)\lambda(\tau) \tag{D.5.18}$$
$$\lambda(t) = -f \tag{D.5.19}$$
$$u^o(\tau) = -\mathbf{K_f}(\tau)\lambda(\tau) \tag{D.5.20}$$
$$g = -\lambda(0) \tag{D.5.21}$$

We see that $u^o(\tau)$ is the output of a linear homogeneous equation. Let $\nu = (t-\tau)$, and define $\mathbf{\Phi}(\nu)$ as the state transition matrix from $\tau = 0$ for the time-varying system having $A - matrix$ equal to $\left[\mathbf{A} - \mathbf{K}_f(t-\nu)^T\mathbf{C}\right]$. Then

$$
\begin{aligned}
\lambda(\tau) &= -\mathbf{\Phi}(t-\tau)^T f \\
\lambda(0) &= -\mathbf{\Phi}(t)^T f \\
u^0(\tau) &= \mathbf{K_f}(\tau)\mathbf{\Phi}(t-\tau)^T f
\end{aligned} \tag{D.5.22}
$$

Hence, the optimal filter satisfies

$$
\begin{aligned}
\hat{z}(t) &= g^T \hat{x}_o + \int_0^t u^o y'(\tau) d\tau \\
&= -\lambda(0)^T \hat{x}_o + \int_0^t f^T \mathbf{\Phi}(t-\tau) \mathbf{K_f}^T(\tau) y'(\tau) d\tau \\
&= f^T \left(\mathbf{\Phi}(t)\hat{x}_o + \int_0^t \mathbf{\Phi}(t-\tau) \mathbf{K_f}^T(\tau) y'(\tau) d\tau \right) \\
&= f^T \hat{x}(t)
\end{aligned} \tag{D.5.23}
$$

where

$$
\hat{x}(t) = \mathbf{\Phi}(t)\hat{x}_o + \int_0^t \mathbf{\Phi}(t-\tau) \mathbf{K_f}^T(\tau) y'(\tau) d\tau \tag{D.5.24}
$$

We then observe that (D.5.24) is actually the solution of the following state space (optimal filter).

$$
\frac{d\hat{x}(t)}{dt} = \left(\mathbf{A} - \mathbf{K_f}^T(t)\mathbf{C} \right) \hat{x}(t) + \mathbf{K_f}^T(t) y'(t) \tag{D.5.25}
$$

$$
\hat{x}(0) = \hat{x}_o \tag{D.5.26}
$$

$$
\hat{z}(t) = f^T x(t) \tag{D.5.27}
$$

We see that the final solution depends on f only through (D.5.27). Thus, as predicted, (D.5.25), (D.5.26) can be used to generate an optimal estimate of any linear combination of states.

Of course, the optimal filter (D.5.25) is identical to that given in (22.10.23)

All of the properties of the optimal filter follow by analogy from the (dual) optimal linear regulator. In particular, we observe that (D.5.16) and (D.5.17) are a CTDRE and its boundary condition, respectively. The only difference is that, in the optimal-filter case, this equation has to be solved forward in time. Also, (D.5.16) has an associated CTARE, given by

$$\mathbf{Q} - \boldsymbol{\Sigma}_\infty \mathbf{C}^T \mathbf{R}^{-1} \mathbf{C} \boldsymbol{\Sigma}_\infty + \boldsymbol{\Sigma}_\infty \mathbf{A}^T + \mathbf{A} \boldsymbol{\Sigma}_\infty = \mathbf{0} \tag{D.5.28}$$

Thus, the existence, uniqueness, and properties of stabilizing solutions for (D.5.16) and (D.5.28) satisfy the same conditions as the corresponding Riccati equations for the optimal regulator.