Appendix C

RESULTS FROM ANALYTIC FUNCTION THEORY

C.1 Introduction

This appendix summarizes key results from analytic function theory leading to the Cauchy Integral formula and its consequence, the Poisson–Jensen formula.

C.2 Independence of Path

Consider functions of two independent variables, x and y. (The reader can think of x as the real axis and y as the imaginary axis.)

Let $P(x,y)$ and $Q(x,y)$ be two functions of x and y, continuous in some domain D. Say we have a curve C in D, described by the parametric equations

$$x = f_1(t), \qquad y = f_2(t) \tag{C.2.1}$$

We can then define the following line integrals along the path C from point A to point B inside D.

$$\int_A^B P(x,y)dx = \int_{t_1}^{t_2} P(f_1(t), f_2(t))\frac{df_1(t)}{dt}dt \tag{C.2.2}$$

$$\int_A^B Q(x,y)dy = \int_{t_1}^{t_2} Q(f_1(t), f_2(t))\frac{df_2(t)}{dt}dt \tag{C.2.3}$$

Definition C.1. *The line integral $\int Pdx + Qdy$ is said to be* **independent of the path** *in D if, for every pair of points A and B in D, the value of the integral is independent of the path followed from A to B.*

□□□

We then have the following result.

Theorem C.1. *If $\int P dx + Q dy$ is independent of the path in D, then there exists a function $F(x, y)$ in D such that*

$$\frac{\partial F}{\partial x} = P(x, y); \qquad \frac{\partial F}{\partial y} = Q(x, y) \tag{C.2.4}$$

hold throughout D. Conversely, if a function $F(x, y)$ can be found such that (C.2.4) hold, then $\int P dx + Q dy$ is independent of the path.

Proof

Suppose that the integral is independent of the path in D. Then, choose a point (x_0, y_0) in D and let $F(x, y)$ be defined as follows

$$F(x, y) = \int_{x_0, y_0}^{x, y} P dx + Q dy \tag{C.2.5}$$

where the integral is taken on an arbitrary path in D joining (x_0, y_0) and (x, y). Because the integral is independent of the path, the integral does indeed depend only on (x, y) and defines the function $F(x, y)$. It remains to establish (C.2.4).

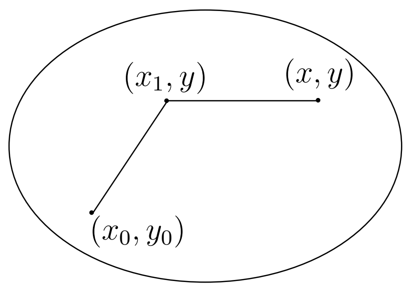

Figure C.1. Integration path

For a particular (x, y) in D, choose (x_1, y) so that $x_1 \neq x$ and so that the line segment from (x_1, y) to (x, y) in D is as shown in Figure C.1. Because of independence of the path,

$$F(x, y) = \int_{x_0, y_0}^{x_1, y} (P dx + Q dy) + \int_{x_1, y}^{x, y} (P dx + Q dy) \tag{C.2.6}$$

We think of x_1 and y as being fixed while (x, y) may vary along the horizontal line segment. Thus $F(x, y)$ is being considered as function of x. The first integral on the right-hand side of (C.2.6) is then independent of x.

Hence, for fixed y, we can write

$$F(x,y) = \text{constant} + \int_{x_1}^{x} P(x,y)dx \tag{C.2.7}$$

The fundamental theorem of Calculus now gives

$$\frac{\partial F}{\partial x} = P(x,y) \tag{C.2.8}$$

A similar argument shows that

$$\frac{\partial F}{\partial y} = Q(x,y) \tag{C.2.9}$$

Conversely, let (C.2.4) hold for some F. Then, with t as a parameter,

$$F(x,y) = \int_{x_1,y_1}^{x_2,y_2} Pdx + Qdy = \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} \right) dt \tag{C.2.10}$$

$$= \int_{t_1}^{t_2} \frac{\partial F}{\partial t} dt \tag{C.2.11}$$

$$= F(x_2,y_2) - F(x_1,y_1) \tag{C.2.12}$$

□□□

Theorem C.2. *If the integral $\int Pdx + Qdy$ is independent of the path in D, then*

$$\oint Pdx + Qdy = 0 \tag{C.2.13}$$

on every closed path in D. Conversely if (C.2.13) holds for every simple closed path in D, then $\int Pdx + Qdy$ is independent of the path in D.

Proof

Suppose that the integral is independent of the path. Let C be a simple closed path in D, and divide C into arcs $\vec{AB}$ and $\vec{BA}$ as in Figure C.2.

$$\oint_C (Pdx + Qdy) = \int_{AB} Pdx + Qdy + \int_{BA} Pdx + Qdy \tag{C.2.14}$$

$$= \int_{AB} Pdx + Qdy - \int_{AB} Pdx + Qdy \tag{C.2.15}$$

The converse result is established by reversing the above argument.

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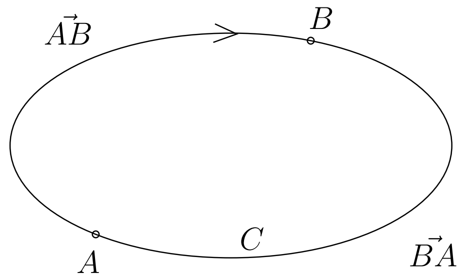

Figure C.2. Integration path

Theorem C.3. *If $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives in D and $\int Pdx + Qdy$ is independent of the path in D, then*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{in } D \tag{C.2.16}$$

Proof

By Theorem C.1, there exists a function F such that (C.2.4) holds. Equation (C.2.16) follows by partial differentiation.

□□□

Actually, we will be particularly interested in the converse to Theorem C.3. However, this holds under slightly more restrictive assumptions, namely a simply connected domain.

C.3 Simply Connected Domains

Roughly speaking, a domain D is simply connected if it has no holes. More precisely, D is simply connected if, for every simple closed curve C in D, the region R enclosed by C lies wholly in D. For simply connected domains we have the following:

Theorem C.4 (Green's theorem). *Let D be a simply connected domain, and let C be a piecewise-smooth simple closed curve in D. Let $P(x, y)$ and $Q(x, y)$ be functions that are continuous and that have continuous first partial derivatives in D. Then*

$$\oint (Pdx + Qdy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \tag{C.3.1}$$

where R is the region bounded by C.

Proof

We first consider a simple case in which R is representable in both of the forms:

$$f_1(x) \le f_2(x) \qquad \text{for } a \le x \le b \tag{C.3.2}$$
$$g_1(y) \le g_2(y) \qquad \text{for } c \le y \le d \tag{C.3.3}$$

Then

$$\int\int_R \frac{\partial P}{\partial y} dxdy = \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y} dxdy \tag{C.3.4}$$

One can now integrate to achieve

$$\int\int_R \frac{\partial P}{\partial y} dxdy = \int_a^b [P(x, f_2(x)) - P(x, f_1(x))]dx \tag{C.3.5}$$
$$= \int_a^b P(x, f_2(x))dx - \int_a^b P(x, f_1(x))dx \tag{C.3.6}$$
$$= \oint_C P(x, y)dx \tag{C.3.7}$$

By a similar argument,

$$\int\int_R \frac{\partial Q}{\partial x} dxdy = \oint_C Q(x, y)dy \tag{C.3.8}$$

For more complex regions, we decompose into simple regions as above. The result then follows.

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We then have the following converse to Theorem C.3.

Theorem C.5. *Let $P(x, y)$ and $Q(x, y)$ have continuous derivatives in D and let D be simply connected. If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\oint Pdx + Qdy$ is independent of path in D.*

Proof

Suppose that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{in } D \tag{C.3.9}$$

Then, by Green's Theorem (Theorem C.4),

$$\oint_c Pdx + Qdy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = 0 \tag{C.3.10}$$

□□□

C.4 Functions of a Complex Variable

In the sequel, we will let $z = x + jy$ denote a complex variable. Note that z is not the argument in the Z-transform, as used at other points in the book. Also, a function $f(z)$ of a complex variable is equivalent to a function $\bar{f}(x, y)$. This will have real and imaginary parts $u(x, y)$ and $v(x, y)$ respectively.

We can thus write

$$f(z) = u(x, y) + jv(x, y) \tag{C.4.1}$$

Note that we also have

$$\begin{aligned}\int_C f(z)dz &= \int_C (u(x, y) + jv(x, y))(dx + jdy) \\ &= \int_C u(x, y)dx - \int_C v(x, y)dy + j \left\{ \int_C u(x, y)dy + \int_C v(x, y)dx \right\}\end{aligned}$$

We then see that the previous results are immediately applicable to the real and imaginary parts of integrals of this type.

C.5 Derivatives and Differentials

Let $w = f(z)$ be a given complex function of the complex variable z. Then w is said to have a derivative at z_0 if

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{C.5.1}$$

exists and is independent of the direction of Δz. We denote this limit, when it exists, by $f'(z_0)$.

C.6 Analytic Functions

Definition C.2. *A function $f(z)$ is said to be analytic in a domain D if f has a continuous derivative in D.*

□□□

Theorem C.6. *If $w = f(z) = u + jv$ is analytic in D, then u and v have continuous partial derivatives satisfying the Cauchy-Riemman conditions.*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{C.6.1}$$

Furthermore

$$\frac{\partial w}{\partial z} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + j\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - j\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y} \tag{C.6.2}$$

Proof

Let z_0 be a fixed point in D and let $\Delta\omega = f(z_0 + \Delta z) - f(z_0)$. Because f is analytic, we have

$$\Delta\omega = \gamma\Delta z + \epsilon\Delta z; \qquad \gamma \stackrel{\triangle}{=} f'(z_0) \tag{C.6.3}$$

where $\gamma = a + jb$ and ϵ goes to zero as $|z_0|$ goes to zero. Then

$$\Delta u + j\Delta v = (a + jb)(\Delta x + j\Delta y) + (\epsilon_1 + j\epsilon_2)(\Delta x + j\Delta y) \tag{C.6.4}$$

So

$$\Delta u = a\Delta x - b\Delta y + \epsilon_1\Delta x - \epsilon_2\Delta y \tag{C.6.5}$$
$$\Delta v = b\Delta x + a\Delta y + \epsilon_2\Delta x + \epsilon_1\Delta y \tag{C.6.6}$$

Thus, in the limit, we can write

$$du = adx - bdy; \qquad dv = bdx - ady \tag{C.6.7}$$

or

$$\frac{\partial u}{\partial x} = a = -\frac{\partial v}{\partial y}; \qquad \frac{\partial u}{\partial y} = -b = -\frac{\partial v}{\partial x} \tag{C.6.8}$$

□□□

Actually, most functions that we will encounter will be analytic, provided the derivative exists. We illustrate this with some examples.

Example C.1. *Consider the function $f(z) = z^2$. Then*

$$f(z) = (x + jy)^2 = x^2 - y^2 + j(2xy) = u + jv \tag{C.6.9}$$

The partial derivatives are

$$\frac{\partial u}{\partial x} = 2x; \qquad \frac{\partial v}{\partial x} = 2y; \qquad \frac{\partial u}{\partial y} = -2y; \qquad \frac{\partial v}{\partial y} = 2x \tag{C.6.10}$$

Hence, the function is clearly analytic.

Example C.2. *Consider* $f(z) = |z|$.

This function is not analytic, because $d|z|$ *is a real quantity and, hence,* $\frac{d|z|}{dz}$ *will depend on the direction of* z.

Example C.3. *Consider a rational function of the form:*

$$W(z) = K\frac{(z-\beta_1)(z-\beta_2)\cdots(z-\beta_m)}{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)} = \frac{N(z)}{D(z)} \tag{C.6.11}$$

$$\frac{\partial W}{\partial z} = \frac{1}{D^2(z)}\left[D(z)\frac{\partial N(z)}{\partial z} - N(z)\frac{\partial D(z)}{\partial z}\right] \tag{C.6.12}$$

These derivatives clearly exist, save when $D = 0$, that is at the poles of $W(z)$.

Example C.4. *Consider the same function* $W(z)$ *defined in (C.6.11). Then*

$$\frac{\partial \ln(W)}{\partial z} = \frac{1}{N(z)D(z)}\left[D(z)\frac{\partial N(z)}{\partial z} - N(z)\frac{\partial D(z)}{\partial z}\right] = \frac{1}{N(z)}\frac{\partial N(z)}{\partial z} - \frac{1}{D(z)}\frac{\partial D(z)}{\partial z} \tag{C.6.13}$$

Hence, $\ln(W(z))$ is analytic, save at the poles and zeros of $W(z)$.

C.7 Integrals Revisited

Theorem C.7 (Cauchy Integral Theorem). *If* $f(z)$ *is analytic in some simply connected domain* D*, then* $\int f(z)dz$ *is independent of path in* D *and*

$$\oint_C f(z)dz = 0 \tag{C.7.1}$$

where C *is a simple closed path in* D.

Proof

This follows from the Cauchy–Riemann conditions together with Theorem C.2.

□□□

We are also interested in the value of integrals in various limiting situations. The following examples cover relevant cases.

We note that if L_C is the length of a simple curve C, then

$$\left|\int_C f(z)dz\right| \leq \max_{z\in C}(|f(z)|)L_C \tag{C.7.2}$$

Example C.5. *Assume that C is a semicircle centered at the origin and having radius R. The path length is then $L_C = \pi R$. Hence,*

- *if $f(z)$ varies as z^{-2}, then $|f(z)|$ on C must vary as R^{-2} – hence, the integral on C vanishes for $R \to \infty$.*
- *if $f(z)$ varies as z^{-1}, then $|f(z)|$ on C must vary as R^{-1} – then, the integral on C becomes a constant as $R \to \infty$.*

Example C.6. *Consider the function $f(z) = \ln(z)$ and an arc of a circle, C, described by $z = \epsilon e^{j\gamma}$ for $\gamma \in [-\gamma_1, \gamma_1]$. Then*

$$I_\epsilon \triangleq \lim_{\epsilon\to 0}\int_C f(z)dz = 0 \tag{C.7.3}$$

This is proven as follows. On C, we have that $f(z) = \ln(\epsilon)$. Then

$$I_\epsilon = \lim_{\epsilon\to 0}\left[(\gamma_2 - \gamma_1)\epsilon\ln(\epsilon)\right] \tag{C.7.4}$$

We then use the fact that $\lim_{|x|\to 0}(x\ln x) = 0$, and the result follows.

Example C.7. *Consider the function*

$$f(z) = \ln\left(1 + \frac{a}{z^n}\right) \qquad n \geq 1 \tag{C.7.5}$$

and a semicircle, C, defined by $z = Re^{j\gamma}$ for $\gamma \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then, if C is followed clockwise,

$$I_R \triangleq \lim_{R\to\infty}\int_C f(z)dz = \begin{cases} 0 & \text{for } n > 1 \\ -j\pi a & \text{for } n = 1 \end{cases} \tag{C.7.6}$$

This is proven as follows.

On C, we have that $z = Re^{j\gamma}$; then

$$I_R = \lim_{R\to\infty} j\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \ln\left(1 + \frac{a}{R^n}e^{-jn\gamma}\right) Re^{j\gamma}d\gamma \tag{C.7.7}$$

We also know that

$$\lim_{|x|\to 0} \ln(1+x) = x \tag{C.7.8}$$

Then

$$I_R = \lim_{R\to\infty} \frac{a}{R^{n-1}} j \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} e^{-j(n-1)\gamma} d\gamma \tag{C.7.9}$$

From this, by evaluation for $n = 1$ and for $n > 1$, the result follows.

□□□

Example C.8. *Consider the function*

$$f(z) = \ln\left(1 + e^{-z\tau}\frac{a}{z^n}\right) \qquad n \geq 1; \quad \tau > 0 \tag{C.7.10}$$

and a semicircle, C, defined by $z = Re^{j\gamma}$ for $\gamma \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then, for clockwise C,

$$I_R \stackrel{\triangle}{=} \lim_{R\to\infty} \int_C f(z)dz = 0 \tag{C.7.11}$$

This is proven as follows.
On C, we have that $z = Re^{j\gamma}$; then

$$I_R = \lim_{R\to\infty} j \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \left[\ln\left(1 + \frac{a}{z^(n+1)}\frac{z}{e^{z\tau}}\right) z\right]_{z=Re^{j\gamma}} d\gamma \tag{C.7.12}$$

We recall that, if τ is a positive real number and $\Re\{z\} > 0$, then

$$\lim_{|z|\to\infty} \frac{z}{e^{z\tau}} = 0 \tag{C.7.13}$$

Moreover, for very large R, we have that

$$\ln\left(1 + \frac{a}{z^{n+1}}\frac{z}{e^{z\tau}}\right) z\Big|_{z=Re^{j\gamma}} \approx \frac{1}{z^n}\frac{z}{e^{z\tau}}\Big|_{z=Re^{j\gamma}} \tag{C.7.14}$$

Thus, in the limit, this quantity goes to zero for all positive n. The result then follows.

□□□

Example C.9. *Consider the function*

$$f(z) = \ln\left(\frac{z-a}{z+a}\right) \tag{C.7.15}$$

and a semicircle, C, defined by $z = Re^{j\gamma}$ for $\gamma \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then, for clockwise C,

$$I_R \triangleq \lim_{R\to\infty} \int_C f(z)dz = j2\pi a \tag{C.7.16}$$

This result is obtained by noting that

$$\ln\left(\frac{z-a}{z+a}\right) = \ln\left(\frac{1-\frac{a}{z}}{1+\frac{a}{z}}\right) = \ln\left(1-\frac{a}{z}\right) - \ln\left(1+\frac{a}{z}\right) \tag{C.7.17}$$

and then applying the result in example C.7.

□□□

Example C.10. *Consider a function of the form*

$$f(z) = \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \dots \tag{C.7.18}$$

and C, an arc of circle $z = Re^{j\theta}$ for $\theta \in [\theta_1, \theta_2]$. Thus, $dz = jzd\theta$, and

$$\int_C \frac{dz}{z} = \int_{\theta_1}^{\theta_2} jd\theta = -j(\theta_2 - \theta_1) \tag{C.7.19}$$

Thus, as $R \to \infty$, we have that

$$\int_C f(z)dz = -ja_{-1}(\theta_2 - \theta_1) \tag{C.7.20}$$

□□□

Example C.11. *Consider, now, $f(z) = z^n$. If the path C is a full circle, centered at the origin and of radius R, then*

$$\oint_C z^n dz = \int_{-\pi}^{\pi} \left(R^n e^{jn\theta}\right) jRe^{j\theta} d\theta \tag{C.7.21}$$

$$= \begin{cases} 0 & \text{for } n \neq -1 \\ -2\pi j & \text{for } n = -1 \text{ (integration clockwise)} \end{cases} \tag{C.7.22}$$

□□□

We can now develop Cauchy's Integral Formula.

Say that $f(z)$ can be expanded as

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots \tag{C.7.23}$$

the a_{-1} is called the residue of $f(z)$ at z_0.

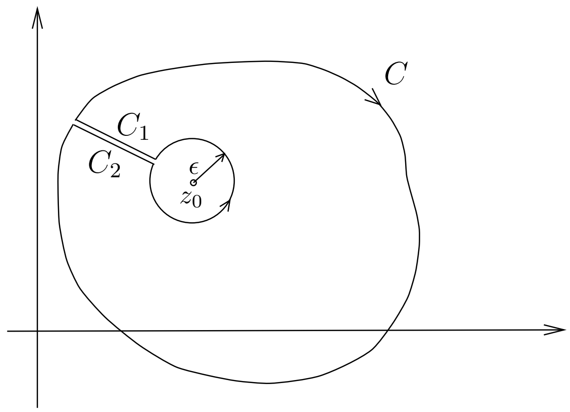

Figure C.3. Path for integration of a function having a singularity

Consider the path shown in Figure C.3. Because $f(z)$ is analytic in a region containing C, we have that the integral around the complete path shown in Figure C.3 is zero. The integrals along C_1 and C_2 cancel. The anticlockwise circular integral around z_0 can be computed by following example C.11 to yield $2\pi j a_{-1}$. Hence, the integral around the outer curve C is minus the integral around the circle of radius ϵ. Thus,

$$\oint_C f(z)dz = -2\pi j a_{-1} \tag{C.7.24}$$

This leads to the following result.

Theorem C.8 (Cauchy's Integral Formula). *Let $g(z)$ be analytic in a region. Let q be a point inside the region. Then $\frac{g(z)}{z-q}$ has residue $g(q)$ at $z = q$, and the integral around any closed contour C enclosing q in a clockwise direction is given by*

$$\oint_C \frac{g(z)}{z - q} dz = -2\pi j g(q) \tag{C.7.25}$$

□□□

We note that the residue of $g(z)$ at an interior point, $z = q$, of a region D can be obtained by integrating $\frac{g(z)}{z-q}$ on the boundary of D. Hence, we can determine the value of an analytic function inside a region by its behaviour on the boundary.

C.8 Poisson and Jensen Integral Formulas

We will next apply the Cauchy Integral formula to develop two related results.

The first result deals with functions that are analytic in the right-half plane (RHP). This is relevant to sensitivity functions in continuous-time systems, where Laplace transforms are used.

The second result deals with functions that are analytic outside the unit disk. This will be a preliminary step to analyzing sensitivity functions in discrete time, on the basis of Z-transforms.

C.8.1 Poisson's Integral for the Half-Plane

Theorem C.9. *Consider a contour C bounding a region D. C is a clockwise contour composed by the imaginary axis and a semicircle to the right, centered at the origin and having radius $R \to \infty$. This contour is shown in Figure C.4. Consider some $z_0 = x_0 + jy_0$ with $x_0 > 0$.*

Let $f(z)$ be a real function of z, analytic inside D and of at least the order of z^{-1}; $f(z)$ satisfies

$$\lim_{|z|\to\infty} |z||f(z)| = \beta \qquad 0 \leq \beta < \infty \qquad z \in D \tag{C.8.1}$$

then

$$f(z_0) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(j\omega)}{j\omega - z_0} d\omega \tag{C.8.2}$$

Moreover, if (C.8.1) is replaced by the weaker condition

$$\lim_{|z|\to\infty} \frac{|f(z)|}{|z|} = 0 \qquad z \in D \tag{C.8.3}$$

then

$$f(z_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(j\omega)\frac{x_0}{x_0^2 + (y_0 - \omega)^2} d\omega \tag{C.8.4}$$

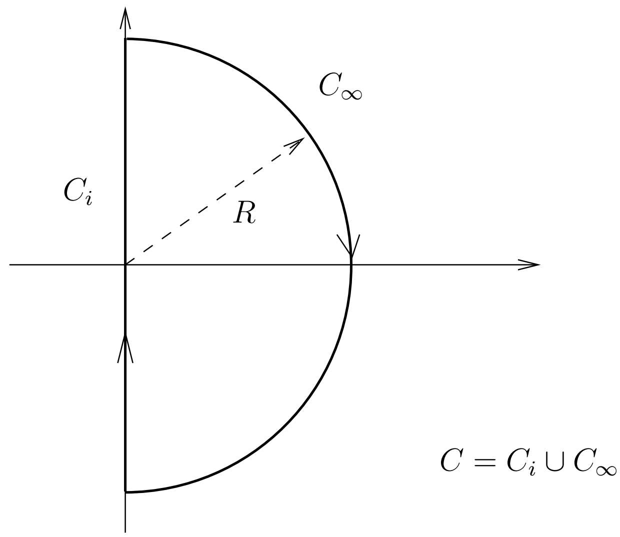

Figure C.4. RHP encircling contour

Proof

Applying Theorem C.8, we have

$$f(z_0) = -\frac{1}{2\pi j}\oint_C \frac{f(z)}{z-z_0}dz = -\frac{1}{2\pi j}\int_{C_i}\frac{f(z)}{z-z_0}dz - \frac{1}{2\pi j}\int_{C_\infty}\frac{f(z)}{z-z_0}dz \quad \text{(C.8.5)}$$

Now, if $f(z)$ satisfies (C.8.1), it behaves like z^{-1} for large $|z|$, i.e., $\frac{f(z)}{z-z_0}$ is like z^{-2}. The integral along C_∞ then vanishes and the result (C.8.2) follows.

To prove (C.8.4) when $f(z)$ satisfies (C.8.3), we first consider z_1, the image of z_0 through the imaginary axis, i.e., $z_1 = -x_0 + jy_0$. Then $\frac{f(z)}{z-z_1}$ is analytic inside D, and, on applying Theorem C.7, we have that

$$0 = -\frac{1}{2\pi j}\oint_C \frac{f(z)}{z-z_1}dz \quad \text{(C.8.6)}$$

By combining equations (C.8.5) and (C.8.6), we obtain

$$f(z_0) = -\frac{1}{2j\pi}\oint_C \left(\frac{f(z)}{z-z_0} - \frac{f(z)}{z-z_1}\right)dz = -\frac{1}{2j\pi}\oint_C f(z)\frac{z_0-z_1}{(z-z_0)(z-z_1)}dz \quad \text{(C.8.7)}$$

Because $C = C_i \cup C_\infty$, the integral over C can be decomposed into the integral along the imaginary axis , C_i, and the integral along the semicircle of infinite radius, C_∞. Because $f(z)$ satisfies (C.8.3), this second integral vanishes, because the factor $\frac{z_0 - z_1}{(z-z_0)(z-z_1)}$ is of order z^{-2} at ∞.

Then

$$f(z_0) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} f(j\omega)\frac{z_0 - z_1}{(j\omega - z_0)(j\omega - z_1)}d\omega \tag{C.8.8}$$

The result follows upon replacing z_0 and z_1 by their real; and imaginary-part decompositions.

□□□

Remark C.1. *One of the functions that satisfies (C.8.3) but does not satisfy (C.8.1) is $f(z) = \ln g(z)$, where $g(z)$ is a rational function of relative degree $n_r \neq 0$. We notice that, in this case,*

$$\lim_{|z|\to\infty}\left[\frac{|\ln g(z)|}{|z|}\right] = \lim_{R\to\infty}\frac{|K||n_r \ln R + jn_r\theta|}{R} = 0 \tag{C.8.9}$$

where K is a finite constant and θ is an angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Remark C.2. *Equation (C.8.4) equates two complex quantities. Thus, it also applies independently to their real and imaginary parts. In particular,*

$$\Re\{f(z_0)\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\Re\{f(j\omega)\}\frac{x_0}{x_0^2 + (y_0 - \omega)^2}d\omega \tag{C.8.10}$$

This observation is relevant to many interesting cases. For instance, when $f(z)$ is as in remark C.1,

$$\Re\{f(z)\} = \ln|g(z)| \tag{C.8.11}$$

For this particular case, and assuming that $g(z)$ is a real function of z, and that $y_0 = 0$, we have that (C.8.10) becomes

$$\ln|g(z_0)| = \frac{1}{\pi}\int_0^{\infty}\ln|g(j\omega)|\frac{2x_0}{x_0^2 + (y_0 - \omega)^2}d\omega \tag{C.8.12}$$

where we have used the conjugate symmetry of $g(z)$.

C.8.2 Poisson–Jensen Formula for the Half-Plane

Lemma C.1. *Consider a function $g(z)$ having the following properties*

(i) $g(z)$ *is analytic on the closed RHP;*

(ii) $g(z)$ *does not vanish on the imaginary axis;*

(iii) $g(z)$ *has zeros in the open RHP, located at* $a_1, a_2, \dots, a_n$*;*

(iv) $g(z)$ *satisfies* $\lim_{|z|\to\infty} \frac{|\ln g(z)|}{|z|} = 0$.

Consider also a point $z_0 = x_0 + jy_0$ *such that* $x_0 > 0$*; then*

$$\ln|g(z_0)| = \sum_{i=1}^{n} \ln\left|\frac{z_0 - a_i}{z_0 + a_i^*}\right| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_0}{x_0^2 + (\omega - y_0)^2} \ln|g(j\omega)|)d\omega \tag{C.8.13}$$

Proof

Let

$$\tilde{g}(z) \triangleq g(z)\prod_{i=1}^{n} \frac{z + a_i^*}{z - a_i} \tag{C.8.14}$$

Then, $\ln \tilde{g}(z)$ is analytic within the closed unit disk. If we now apply Theorem C.9 to $\ln \tilde{g}(z)$, we obtain

$$\ln \tilde{g}(z_0) = \ln g(z_0) + \sum_{i=1}^{n} \ln\left(\frac{z_0 + a_i^*}{z_0 - a_i}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_0}{x_0^2 + (\omega - y_0)^2} \ln \tilde{g}(j\omega)d\omega \tag{C.8.15}$$

We also recall that, if x is any complex number, then $\Re\{\ln x\} = \Re\{\ln|x| + j\angle x\} = \ln|x|$. Thus, the result follows upon equating real parts in the equation above and noting that

$$\ln|\tilde{g}(j\omega)| = \ln|g(j\omega)| \tag{C.8.16}$$

□□□

C.8.3 Poisson's Integral for the Unit Disk

Theorem C.10. *Let $f(z)$ be analytic inside the unit disk. Then, if $z_0 = re^{j\theta}$, with $0 \le r < 1$,*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} P_{1,r}(\theta - \omega) f(e^{j\omega}) d\omega \tag{C.8.17}$$

where $P_{1,r}(x)$ *is the Poisson kernel defined by*

$$P_{\rho,r}(x) \triangleq \frac{\rho^2 - r^2}{\rho^2 - 2r\rho\cos(x) + r^2} \qquad 0 \le r < \rho, \qquad x \in \Re \tag{C.8.18}$$

Proof

Consider the unit circle C. Then, using Theorem C.8, we have that

$$f(z_0) = \frac{1}{2\pi j} \oint_C \frac{f(z)}{z - z_0} dz \tag{C.8.19}$$

Define

$$z_1 \triangleq \frac{1}{r} e^{j\theta} \tag{C.8.20}$$

Because z_1 is outside the region encircled by C, the application of Theorem C.8 yields

$$0 = \frac{1}{2\pi j} \oint_C \frac{f(z)}{z - z_1} dz \tag{C.8.21}$$

Subtracting (C.8.21) from (C.8.19) and changing the variable of integration, we obtain

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{j\omega}) e^{j\omega} \left[\frac{1}{e^{j\omega} - re^{j\theta}} - \frac{r}{re^{j\omega} - e^{j\theta}} \right] d\omega \tag{C.8.22}$$

from which the result follows.

□□□

Consider now a function $g(z)$ which is analytic outside the unit disk. We can then define a function $f(z)$ such that

$$f(z) \triangleq g\left(\frac{1}{z}\right) \tag{C.8.23}$$

Assume that one is interested in obtaining an expression for $g(\zeta_0)$, where $\zeta_0 = re^{j\theta}$, $r > 1$. The problem is then to obtain an expression for $f\left(\frac{1}{\zeta_0}\right)$. Thus, if we define $z_0 \triangleq \frac{1}{\zeta_0} = \frac{1}{r}e^{-j\theta}$, we have, on applying Theorem C.10, that

$$g(\zeta_0) = \frac{1}{2\pi}\int_0^{2\pi} P_{1,\frac{1}{r}}(-\theta-\omega)g(e^{-j\omega})d\omega \tag{C.8.24}$$

where

$$P_{1,\frac{1}{r}}(-\theta-\omega) = \frac{r^2-1}{r^2-2rcos(\theta+\omega)+1} \tag{C.8.25}$$

If, finally, we make the change in the integration variable $\omega = -\nu$, the following result is obtained.

$$g(re^{j\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-1}{r^2-2rcos(\theta-\nu)+1} g(e^{j\nu})d\nu \tag{C.8.26}$$

Thus, Poisson's integral for the unit disk can also be applied to functions of a complex variable which are analytic outside the unit circle.

C.8.4 Poisson–Jensen Formula for the Unit Disk

Lemma C.2. *Consider a function $g(z)$ having the following properties:*

(i) $g(z)$ is analytic on the closed unit disk;

(ii) $g(z)$ does not vanish on the unit circle;

(iii) $g(z)$ has zeros in the open unit disk, located at $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{\bar{n}}$.

Consider also a point $z_0 = re^{j\theta}$ such that $r < 1$; then

$$\ln|g(z_0)| = \sum_{i=1}^{\bar{n}} \ln\left|\frac{z_0-\bar{\alpha}_i}{1-\bar{\alpha}_i^* z_0}\right| + \frac{1}{2\pi}\int_0^{2\pi} P_{1,r}(\theta-\omega)\ln|g(e^{j\omega})|d\omega \tag{C.8.27}$$

Proof

Let

$$\tilde{g}(z) \triangleq g(z)\prod_{i=1}^{n}\frac{1-\bar{\alpha}_i^* z}{z-\bar{\alpha}_i} \tag{C.8.28}$$

Then $\ln \tilde{g}(z)$ is analytic on the closed unit disk. If we now apply Theorem C.10 to $\ln \tilde{g}(z)$, we obtain

$$\ln \tilde{g}(z_0) = \ln g(z_0) + \sum_{i=1}^{n} \ln \left(\frac{1 - \bar{\alpha}_i^* z_0}{z_0 - \bar{\alpha}_i} \right) = \frac{1}{2\pi} \int_0^{2\pi} P_{1,r}(\theta - \omega) \ln \tilde{g}(e^{j\omega}) d\omega \tag{C.8.29}$$

We also recall that, if x is any complex number, then $\ln x = ln|x| + j\angle x$. Thus the result follows upon equating real parts in the equation above and noting that

$$\ln \left| \tilde{g}(e^{j\omega}) \right| = \ln \left| g(e^{j\omega}) \right| \tag{C.8.30}$$

□□□

Theorem C.11 (Jensen's formula for the unit disk). *Let $f(z)$ and $g(z)$ be analytic functions on the unit disk. Assume that the zeros of $f(z)$ and $g(z)$ on the unit disk are $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_{\bar{n}}$ and $\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_{\bar{m}}$ respectively, where none of these zeros lie on the unit circle.*

If

$$h(z) \triangleq z^{\lambda} \frac{f(z)}{g(z)} \qquad \lambda \in \Re \tag{C.8.31}$$

then

$$\frac{1}{2\pi} \int_0^{2\pi} \ln |h(e^{j\omega})| d\omega = \ln \left| \frac{f(0)}{g(0)} \right| + \ln \frac{|\bar{\beta}_1 \bar{\beta}_2 \dots \bar{\beta}_{\bar{m}}|}{|\bar{\alpha}_1 \bar{\alpha}_2 \dots \bar{\alpha}_{\bar{n}}|} \tag{C.8.32}$$

Proof

We first note that $\ln |h(z)| = \lambda \ln |z| + \ln |f(z)| - \ln |g(z)|$. We then apply the Poisson–Jensen formula to $f(z)$ and $g(z)$ at $z_0 = 0$ to obtain

$$P_{1,r}(x) = P_{1,0}(x) = 1; \qquad \ln \left| \frac{z_0 - \bar{\alpha}_i}{1 - \bar{\alpha}_i^* z_0} \right| = \ln |\bar{\alpha}_i|; \qquad \ln \left| \frac{z_0 - \bar{\beta}_i}{1 - \bar{\beta}_i^* z_0} \right| = \ln |\bar{\beta}_i| \tag{C.8.33}$$

We thus have that

$$\ln |f(0)| = \sum_{i=1}^{n} \ln |\bar{\alpha}_i| - \frac{1}{2\pi} \int_0^{2\pi} \ln |f(e^{j\omega})| d\omega \tag{C.8.34}$$

$$\ln |g(0)| = \sum_{i=1}^{n} \ln |\bar{\alpha}_i| - \frac{1}{2\pi} \int_0^{2\pi} \ln |g(e^{j\omega})| d\omega \tag{C.8.35}$$

The result follows upon subtracting equation (C.8.35) from (C.8.34), and noting that

$$\frac{\lambda}{2\pi}\int_0^{2\pi} \ln\left|e^{j\omega}\right| d\omega = 0 \tag{C.8.36}$$

□□□

Remark C.3. *Further insights can be obtained from equation (C.8.32) if we assume that, in* (C.8.31)*, $f(z)$ and $g(z)$ are polynomials;*

$$f(z) \; = \; K_f \prod_{i=1}^{n} (z-\alpha_i) \tag{C.8.37}$$

$$g(z) \; = \; \prod_{i=1}^{n} (z-\beta_i) \tag{C.8.38}$$

then

$$\left|\frac{f(0)}{g(0)}\right| = |K_f| \left|\frac{\prod_{i=1}^{n}\alpha_i}{\prod_{i=1}^{m}\beta_i}\right| \tag{C.8.39}$$

Thus, $\alpha_1, \alpha_2, \ldots \alpha_n$ and $\beta_1, \beta_2, \ldots \beta_m$ are **all** *the zeros and* **all** *the poles of $h(z)$, respectively, that have nonzero magnitude.*

This allows equation (C.8.32) to be rewritten as

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|h(e^{j\omega})| d\omega = \ln|K_f| + \ln\frac{|\alpha'_1\alpha'_2\ldots\alpha'_{nu}|}{|\beta'_1\beta'_2\ldots\beta'_{mu}|} \tag{C.8.40}$$

where $\alpha'_1, \alpha'_2, \ldots \alpha'_{nu}$ and $\beta'_1, \beta'_2, \ldots \beta'_{mu}$ are the zeros and the poles of $h(z)$, respectively, that lie outside the unit circle .

□□□

C.9 Application of the Poisson–Jensen Formula to Certain Rational Functions

Consider the biproper rational function $\bar{h}(z)$ given by

$$\bar{h}(z) = z^{\bar{\lambda}} \frac{\bar{f}(z)}{\bar{g}(z)} \tag{C.9.1}$$

$\bar{\lambda}$ is a integer number, and $\bar{f}(z)$ and $\bar{g}(z)$ are polynomials of degrees m_f and m_g, respectively. Then, due to the biproperness of $\bar{h}(z)$, we have that $\bar{\lambda} + m_f = m_g$.

Further assume that

(i) $\bar{g}(z)$ has no zeros outside the open unit disk,

(ii) $\bar{f}(z)$ does not vanish on the unit circle, and

(iii) $\bar{f}(z)$ vanishes outside the unit disk at $\beta_1, \beta_2, \ldots, \beta_m$.

Define

$$h(z) = \frac{f(z)}{g(z)} \triangleq \bar{h}\left(\frac{1}{z}\right) \tag{C.9.2}$$

where $f(z)$ and $g(z)$ are polynomials.

Then it follows that

(i) $g(z)$ has no zeros in the closed unit disk;

(ii) $f(z)$ does not vanish on the unit circle;

(iii) $f(z)$ vanishes in the open unit disk at $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m$, where $\bar{\beta}_i = \beta_i^{-1}$ for $i = 1, 2, \ldots, \bar{\beta}_m$;

(iv) $h(z)$ is analytic in the closed unit disk;

(v) $h(z)$ does not vanish on the unit circle;

(vi) $h(z)$ has zeros in the open unit disk, located at $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m$.

We then have the following result

Lemma C.3. *Consider the function $h(z)$ defined in (C.9.2) and a point $z_0 = re^{j\theta}$ such that $r < 1$; then*

$$\ln |h(z_0)| = \sum_{i=1}^{\bar{m}} \ln \left| \frac{z_0 - \bar{\beta}_i}{1 - \bar{\beta}_i^* z_0} \right| + \frac{1}{2\pi} \int_0^{2\pi} P_{1,r}(\theta - \omega) \ln |h(e^{j\omega})| d\omega \tag{C.9.3}$$

where $P_{1,r}$ is the Poisson kernel defined in (C.8.18).

Proof

This follows from a straightforward application of Lemma C.2.

□□□

C.10 Bode's Theorems

We will next review some fundamental results due to Bode.

Theorem C.12 (Bode integral in the half plane). *Let $l(z)$ be a proper real, rational function of relative degree n_r. Define*

$$g(z) \triangleq (1 + l(z))^{-1} \tag{C.10.1}$$

and assume that $g(z)$ has neither poles nor zeros in the closed RHP. Then

$$\int_0^\infty \ln |g(j\omega)| d\omega = \begin{cases} 0 & \text{for } n_r > 1 \\ -\kappa\frac{\pi}{2} & \text{for } n_r = 1 \quad \text{where} \quad \kappa \triangleq \lim_{z\to\infty} zl(z) \end{cases} \tag{C.10.2}$$

Proof

Because $\ln g(z)$ is analytic in the closed RHP,

$$\oint_C \ln g(z) dz = 0 \tag{C.10.3}$$

where $C = C_i \cup C_\infty$ is the contour defined in Figure C.4.

Then

$$\oint_C \ln g(z) dz = j \int_{-\infty}^{\infty} \ln g(j\omega) d\omega - \int_{C_\infty} \ln(1 + l(z)) dz \tag{C.10.4}$$

For the first integral on the right-hand side of equation (C.10.4), we use the conjugate symmetry of $g(z)$ to obtain

$$\int_{-\infty}^{\infty} \ln g(j\omega) d\omega = 2 \int_0^\infty \ln |g(j\omega)| d\omega \tag{C.10.5}$$

For the second integral, we notice that, on C_∞, $l(z)$ can be approximated by

$$\frac{a}{z^{n_r}} \tag{C.10.6}$$

The result follows upon using example C.7 and noticing that $a = \kappa$ for $n_r = 1$.

□□□

Remark C.4. *If $g(z) = (1 + e^{-z\tau} l(z))^{-1}$ for $\tau > 0$, then result (C.10.9) becomes*

$$\int_0^{\infty} \ln|g(j\omega)|d\omega = 0 \qquad \forall n_r > 0 \tag{C.10.7}$$

The proof of (C.10.7) follows along the same lines as those of Theorem C.12 and by using the result in example C.8.

Theorem C.13 (Modified Bode integral). *Let $l(z)$ be a proper real, rational function of relative degree n_r. Define*

$$g(z) \triangleq (1 + l(z))^{-1} \tag{C.10.8}$$

Assume that $g(z)$ is analytic in the closed RHP and that it has q zeros in the open RHP, located at $\zeta_1, \zeta_2, \dots, \zeta_q$ with $\Re(\zeta_i) > 0$. Then

$$\int_0^{\infty} \ln|g(j\omega)|d\omega = \begin{cases} \pi \sum_{i=1}^{q} \zeta_i & \text{for } n_r > 1 \\ -\kappa\frac{\pi}{2} + \pi \sum_{i=1}^{q} \zeta_i & \text{for } n_r = 1 \quad \text{where} \quad \kappa \triangleq \lim_{z\to\infty} zl(z) \end{cases} \tag{C.10.9}$$

Proof

We first notice that $\ln g(z)$ is no longer analytic on the RHP. We then define

$$\tilde{g}(z) \triangleq g(z) \prod_{i=1}^{q} \frac{z + \zeta_i}{z - \zeta_i} \tag{C.10.10}$$

Thus, $\ln \tilde{g}(z)$ is analytic in the closed RHP. We can then apply Cauchy's integral in the contour C described in Figure C.4 to obtain

$$\oint_C \ln \tilde{g}(z)dz = 0 = \oint_C \ln g(z)dz + \sum_{i=1}^{q} \oint_C \ln \frac{z + \zeta_i}{z - \zeta_i}dz \tag{C.10.11}$$

The first integral on the right-hand side can be expressed as

$$\oint_C \ln g(z)dz = 2j \int_0^{\infty} \ln|g(j\omega)|d\omega + \int_{C_\infty} \ln g(z)dz \tag{C.10.12}$$

where, by using example C.7.

$$\int_{C_\infty} \ln g(z)dz = \begin{cases} 0 & \text{for } n_r > 1 \\ j\kappa\pi & \text{for } n_r = 1 \quad \text{where} \quad \kappa \triangleq \lim_{z\to\infty} zl(z) \end{cases} \tag{C.10.13}$$

The second integral on the right-hand side of equation (C.10.11) can be computed as follows:

$$\oint_C \ln \frac{z+\zeta_i}{z-\zeta_i} dz = j \int_{-\infty}^{\infty} \ln \frac{j\omega+\zeta_i}{j\omega-\zeta_i} d\omega + \int_{C_\infty} \ln \frac{z+\zeta_i}{z-\zeta_i} dz \tag{C.10.14}$$

We note that the first integral on the right-hand side is zero, and by using example C.9, the second integral is equal to $-2j\pi\zeta_i$. Thus, the result follows.

□□□

Remark C.5. *Note that $g(z)$ is a real function of z, so*

$$\sum_{i=1}^{q} \zeta_i = \sum_{i=1}^{q} \Re\{\zeta_i\} \tag{C.10.15}$$

□□□

Remark C.6. *If $g(z) = \left(1 + e^{-z\tau} l(z)\right)^{-1}$ for $\tau > 0$, then the result (C.10.9) becomes*

$$\int_0^{\infty} \ln |g(j\omega)| d\omega = \pi \sum_{i=1}^{q} \Re\{\zeta_i\} \qquad \forall n_r > 0 \tag{C.10.16}$$

The proof of (C.10.16) follows along the same lines as those of Theorem C.13 and by using the result in example C.8.

Remark C.7. *The Poisson, Jensen, and Bode formulae assume that a key function is analytic, not only inside a domain D, but also on its border C. Sometimes, there may exist singularities on C. These can be dealt with by using an infinitesimal circular indentation in C, constructed so as to leave the singularity outside D. For the functions of interest to us, the integral along the indentation vanishes. This is illustrated in example C.6 for a logarithmic function, when D is the right-half plane and there is a singularity at the origin.*

□□□