Appendix B

SMITH–MCMILLAN FORMS

B.1 Introduction

Smith–McMillan forms correspond to the underlying structures of natural MIMO transfer-function matrices. The key ideas are summarized below.

B.2 Polynomial Matrices

Multivariable transfer functions depend on polynomial matrices. There are a number of related terms that are used. Some of these are introduced here:

Definition B.1. *A matrix* $\mathbf{\Pi}(s) = [p_{ik}(s)] \in \mathbb{R}^{n_1 \times n_2}$ *is a* ***polynomial matrix*** *if* $p_{ik}(s)$ *is a polynomial in* s*, for* $i = 1, 2, \ldots, n_1$ *and* $k = 1, 2, \ldots, n_2$.

Definition B.2. *A polynomial matrix* $\mathbf{\Pi}(s)$ *is said to be a* ***unimodular matrix*** *if its determinant is a constant. Clearly, the inverse of a unimodular matrix is also a unimodular matrix.*

Definition B.3. *An* ***elementary operation*** *on a polynomial matrix is one of the following three operations:*

(eo1) interchange of two rows or two columns;

(eo2) multiplication of one row or one column by a constant;

(eo3) addition of one row (column) to another row (column) times a polynomial.

Definition B.4. *A left (right)* ***elementary matrix*** *is a matrix such that, when it multiplies from the left (right) a polynomial matrix, then it performs a row (column) elementary operation on the polynomial matrix. All elementary matrices are unimodular.*

Definition B.5. *Two polynomial matrices* $\mathbf{\Pi}_1(s)$ *and* $\mathbf{\Pi}_2(s)$ *are* ***equivalent matrices****, if there exist sets of left and right elementary matrices,* $\{\mathbf{L}_1(s), \mathbf{L}_2(s), \ldots, \mathbf{L}_{k1}\}$ *and* $\{\mathbf{R}_1(s), \mathbf{R}_2(s), \ldots, \mathbf{R}_{k2}\}$*, respectively, such that*

$$\mathbf{\Pi}_1(s) = \mathbf{L}_{k1}(s) \cdots \mathbf{L}_2(s)\mathbf{L}_1\mathbf{\Pi}_2(s)\mathbf{R}_1(s)\mathbf{R}_2(s) \cdots \mathbf{R}_{k2} \tag{B.2.1}$$

Definition B.6. *The* ***rank of a polynomial matrix*** *is the rank of the matrix almost everywhere in s. The definition implies that the rank of a polynomial matrix is independent of the argument.*

Definition B.7. *Two polynomial matrices $\mathbf{V}(s)$ and $\mathbf{W}(s)$ having the same number of columns (rows) are* ***right (left) coprime*** *if all common right (left) factors are unimodular matrices.*

Definition B.8. *The degree ∂_{ck} (∂_{rk}) of the k^{th} column (row) $[\mathbf{V}(s)]_{*k}$ ($[\mathbf{V}(s)]_{k*}$) of a polynomial matrix $\mathbf{V}(s)$ is the degree of highest power of s in that column (row).*

Definition B.9. *A polynomial matrix $\mathbf{V}(s) \in \mathbb{C}^{m\times m}$ is column proper if*

$$\lim_{s\to\infty} \det(\mathbf{V}(s)\,\mathrm{diag}\left(s^{-\partial_{c1}}, s^{-\partial_{c2}}, \dots , s^{-\partial_{cm}}\right)) \tag{B.2.2}$$

has a finite, nonzero value.

Definition B.10. *A polynomial matrix $\mathbf{V}(s) \in \mathbb{C}^{m\times m}$ is row proper if*

$$\lim_{s\to\infty} \det(\mathrm{diag}\left(s^{-\partial_{r1}}, s^{-\partial_{r2}}, \dots , s^{-\partial_{rm}}\right)\mathbf{V}(s)) \tag{B.2.3}$$

has a finite, nonzero value.

B.3 Smith Form for Polynomial Matrices

Using the above notation, we can manipulate polynomial matrices in ways that mirror the ways we manipulate matrices of reals. For example, the following result describes a diagonal form for polynomial matrices.

Theorem B.1 (Smith form). *Let $\mathbf{\Pi}(s)$ be a $m_1 \times m_2$ polynomial matrix of rank r; then $\mathbf{\Pi}(s)$ is equivalent to either a matrix $\mathbf{\Pi}_f(s)$ (for $m_1 < m_2$) or to a matrix $\mathbf{\Pi}_c(s)$ (for $m_2 < m_1$), with*

$$\mathbf{\Pi}_f(s) = \begin{bmatrix}\mathbf{E}(s) & \Theta_f\end{bmatrix}; \qquad \mathbf{\Pi}_c(s) = \begin{bmatrix}\mathbf{E}(s)\\ \Theta_c\end{bmatrix} \tag{B.3.1}$$

$$\mathbf{E}(s) = \mathrm{diag}(\overline{\epsilon}_1(s), \dots , \overline{\epsilon}_r(s), 0, \dots , 0) \tag{B.3.2}$$

where Θ_f and Θ_c are matrices with all their elements equal to zero.

Furthermore $\overline{\epsilon}_i(s)$ are monic polynomials for $i = 1, 2, \dots , r$, such that $\overline{\epsilon}_i(s)$ is a factor in $\overline{\epsilon}_{i+1}(s)$, i.e. $\overline{\epsilon}_i(s)$ divides $\overline{\epsilon}_{i+1}(s)$.

If $m_1 = m_2$, then $\mathbf{\Pi}(s)$ is equivalent to the square matrix $\mathbf{E}(s)$.

Proof (by construction)

(i) By performing row and column interchange operations on $\mathbf{\Pi}(s)$, bring to position (1,1) the least degree polynomial entry in $\mathbf{\Pi}(s)$. Say this minimum degree is ν_1

(ii) *Using elementary operation (e03) (see definition B.3), reduce the term in the position (2,1) to degree* $\nu_2 < \nu_1$*. If the term in position (2,1) becomes zero, then go to the next step, otherwise, interchange rows 1 and 2 and repeat the procedure until the term in position (2,1) becomes zero.*

(iii) *Repeat step (ii) with the other elements in the first column.*

(iv) *Apply the same procedure to all the elements but the first one in the first row.*

(v) *Go back to step (ii) if nonzero entries due to step (iv) appear in the first column. Notice that the degree of the entry (1,1) will fall in each cycle, until we finally end up with a matrix which can be partitioned as*

$$\mathbf{\Pi}(s) = \begin{bmatrix} \overline{\pi}_{11}^{(j)}(s) & 0 \quad 0 & \dots & 0 \quad 0 \\ 0 & & & \\ 0 & & & \\ \vdots & & \mathbf{\Pi_j}(s) & \\ 0 & & & \\ 0 & & & \end{bmatrix} \tag{B.3.3}$$

where $\overline{\pi}_{11}^{(j)}(s)$ *is a monic polynomial.*

(vi) *If there is an element of* $\mathbf{\Pi_j}(s)$ *which is of lesser degree than* $\overline{\pi}_{11}^{(j)}(s)$*, then add the column where this element is to the first column and repeat steps (ii) to (v). Do this until the form* (B.3.3) *is achieved with* $\overline{\pi}_{11}^{(j)}(s)$ *of less or, at most, equal degree to that of every element in* $\mathbf{\Pi_j}(s)$*. This will yield further reduction in the degree of the entry in position (1,1).*

(vii) *Make* $\overline{\epsilon}_1(s) = \overline{\pi}_{11}^{(j)}(s)$*.*

(viii) *Repeat the procedure from steps (i) through (viii) to matrix* $\mathbf{\Pi_j}(s)$*.*

Actually the polynomials $\overline{\epsilon}_i(s)$ in the above result can be obtained in a direct fashion, as follows:

(i) Compute all minor determinants of $\mathbf{\Pi}(s)$.

(ii) Define $\chi_i(s)$ as the (monic) greatest common divisor (g.c.d.) of all $i \times i$ minor determinants of $\mathbf{\Pi}(s)$. Make $\chi_0(s) = 1$.

(iii) Compute the polynomials $\overline{\epsilon}_i(s)$ as

$$\overline{\epsilon}_i(s) = \frac{\chi_i(s)}{\chi_{i-1}(s)} \tag{B.3.4}$$

B.4 Smith–McMillan Form for Rational Matrices

A straightforward application of Theorem B.1 leads to the following result, which gives a diagonal form for a rational transfer-function matrix:

Theorem B.2 (Smith–McMillan form). *Let* $\mathbf{G}(s) = [G_{ik}(s)]$ *be an* $m \times m$ *matrix transfer function, where* $G_{ik}(s)$ *are rational scalar transfer functions:*

$$\mathbf{G}(s) = \frac{\mathbf{\Pi}(s)}{D_G(s)} \tag{B.4.1}$$

where $\mathbf{\Pi}(s)$ *is an* $m \times m$ *polynomial matrix of rank* r *and* $D_G(s)$ *is the least common multiple of the denominators of all elements* $G_{ik}(s)$.

Then, $\mathbf{G}(s)$ *is equivalent to a matrix* $\mathbf{M}(s)$, *with*

$$\mathbf{M}(s) = \text{diag}\left(\frac{\epsilon_1(s)}{\delta_1(s)}, \ldots, \frac{\epsilon_r(s)}{\delta_r(s)}, 0, \ldots, 0\right) \tag{B.4.2}$$

where $\{\epsilon_i(s), \delta_i(s)\}$ *is a pair of monic and coprime polynomials for* $i = 1, 2, \ldots, r$.

Furthermore, $\epsilon_i(s)$ *is a factor of* $\epsilon_{i+1}(s)$ *and* $\delta_i(s)$ *is a factor of* $\delta_{i-1}(s)$.

Proof

We write the transfer-function matrix as in (B.4.1). We then perform the algorithm outlined in Theorem B.1 to convert $\mathbf{\Pi}(s)$ *to Smith normal form. Finally, canceling terms for the denominator* $D_G(s)$ *leads to the form given in (B.4.2).*

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We use the symbol $\mathbf{G}^{SM}(s)$ to denote $\mathbf{M}(s)$, which is the *Smith–McMillan form* of the transfer-function matrix $\mathbf{G}(s)$.

We illustrate the formula of the Smith–McMillan form by a simple example.

Example B.1. *Consider the following transfer-function matrix*

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{4}{(s+1)(s+2)} & \dfrac{-1}{s+1} \\ \\ \dfrac{2}{s+1} & \dfrac{-1}{2(s+1)(s+2)} \end{bmatrix} \tag{B.4.3}$$

We can then express $\mathbf{G}(s)$ *in the form (B.4.1):*

$$\mathbf{G}(s) = \frac{\mathbf{\Pi}(s)}{D_G(s)}; \quad \mathbf{\Pi}(s) = \begin{bmatrix} 4 & -(s+2) \\ 2(s+2) & -\dfrac{1}{2} \end{bmatrix}; \quad D_G(s) = (s+1)(s+2) \tag{B.4.4}$$

The polynomial matrix $\mathbf{\Pi}(s)$ can be reduced to the Smith form defined in Theorem B.1. To do that, we first compute its greatest common divisors:

$$\chi_0 = 1 \tag{B.4.5}$$

$$\chi_1 = gcd\left\{4; -(s+2); 2(s+2); -\frac{1}{2}\right\} = 1 \tag{B.4.6}$$

$$\chi_2 = gcd\{2s^2+8s+6\} = s^2+4s+3 = (s+1)(s+3) \tag{B.4.7}$$

This leads to

$$\bar{\epsilon}_1 = \frac{\chi_1}{\chi_0} = 1; \qquad \bar{\epsilon}_2 = \frac{\chi_2}{\chi_1} = (s+1)(s+3) \tag{B.4.8}$$

From here, the Smith–McMillan form can be computed to yield

$$\mathbf{G^{SM}}(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & 0 \\ 0 & \frac{s+3}{s+2} \end{bmatrix} \tag{B.4.9}$$

B.5 Poles and Zeros

The Smith–McMillan form can be utilized to give an unequivocal definition of poles and zeros in the multivariable case. In particular, we have:

Definition B.11. *Consider a transfer-function matrix , $\mathbf{G}(s)$.*

(i) *$p_z(s)$ and $p_p(s)$ are said to be the **zero polynomial** and the **pole polynomial** of $\mathbf{G}(s)$, respectively, where*

$$p_z(s) \stackrel{\triangle}{=} \epsilon_1(s)\epsilon_2(s)\cdots\epsilon_r(s); \qquad p_p(s) \stackrel{\triangle}{=} \delta_1(s)\delta_2(s)\cdots\delta_r(s) \tag{B.5.1}$$

and where $\epsilon_1(s)$, $\epsilon_2(s)$, ..., $\epsilon_r(s)$ and $\delta_1(s)$, $\delta_2(s)$, ..., $\delta_r(s)$ are the polynomials in the Smith–McMillan form, $\mathbf{G}^{SM}(s)$ of $\mathbf{G}(s)$.

Note that $p_z(s)$ and $p_p(s)$ are monic polynomials.

(ii) *The **zeros** of the matrix $\mathbf{G}(s)$ are defined to be the roots of $p_z(s)$, and the **poles** of $\mathbf{G}(s)$ are defined to be the roots of $p_p(s)$.*

(iii) *The **McMillan degree** of $\mathbf{G}(s)$ is defined as the degree of $p_p(s)$.*

In the case of square plants (same number of inputs as outputs), it follows that $\det[\mathbf{G}(s)]$ is a simple function of $p_z(s)$ and $p_p(s)$. Specifically, we have

$$\det[\mathbf{G}(s)] = K_\infty \frac{p_z(s)}{p_p(s)} \tag{B.5.2}$$

Note, however, that $p_z(s)$ and $p_p(s)$ are not necessarily coprime. Hence, the scalar rational function $\det[\mathbf{G}(s)]$ is not sufficient to determine all zeros and poles of $\mathbf{G}(s)$. However, the relative degree of $\det[\mathbf{G}(s)]$ is equal to the difference between the number of poles and the number of zeros of the MIMO transfer-function matrix.

B.6 Matrix Fraction Descriptions (MFD)

A model structure that is related to the Smith–McMillan form is that of a **matrix fraction description** (MFD). There are two types, namely a right matrix fraction description (RMFD) and a left matrix fraction description (LMFD).

We recall that a matrix $\mathbf{G}(s)$ and its Smith-McMillan form $\mathbf{G}^{\mathbf{SM}}(s)$ are equivalent matrices. Thus, there exist two unimodular matrices, $\mathbf{L}(s)$ and $\mathbf{R}(s)$, such that

$$\mathbf{G}^{\mathbf{SM}}(s) = \mathbf{L}(s)\mathbf{G}(s)\mathbf{R}(s) \tag{B.6.1}$$

This implies that if $\mathbf{G}(s)$ is an $m \times m$ proper transfer-function matrix, then there exist a $m \times m$ matrix $\tilde{\mathbf{L}}(s)$ and an $m \times m$ matrix $\tilde{\mathbf{R}}(s)$, such as

$$\mathbf{G}(s) = \tilde{\mathbf{L}}(s)\mathbf{G}^{\mathbf{SM}}(s)\tilde{\mathbf{R}}(s) \tag{B.6.2}$$

where $\tilde{\mathbf{L}}(s)$ and $\tilde{\mathbf{R}}(s)$ are, for example, given by

$$\tilde{\mathbf{L}}(s) = [\mathbf{L}(s)]^{-1}; \qquad \tilde{\mathbf{R}}(s) = [\mathbf{R}(s)]^{-1} \tag{B.6.3}$$

We next define the following two matrices:

$$\mathbf{N}(s) \stackrel{\triangle}{=} \mathrm{diag}(\epsilon_1(s), \ldots, \epsilon_r(s), 0, \ldots, 0) \tag{B.6.4}$$

$$\mathbf{D}(s) \stackrel{\triangle}{=} \mathrm{diag}(\delta_1(s), \ldots, \delta_r(s), 1, \ldots, 1) \tag{B.6.5}$$

where $\mathbf{N}(s)$ and $\mathbf{D}(s)$ are $m \times m$ matrices. Hence, $\mathbf{G}^{\mathbf{SM}}(s)$ can be written as

$$\mathbf{G}^{\mathbf{SM}}(s) = \mathbf{N}(s)[\mathbf{D}(s)]^{-1} \tag{B.6.6}$$

Combining (B.6.2) and (B.6.6), we can write

$$\mathbf{G}(s) = \tilde{\mathbf{L}}(s)\mathbf{N}(s)[\mathbf{D}(s)]^{-1}\tilde{\mathbf{R}}(s) = [\tilde{\mathbf{L}}(s)\mathbf{N}(s)][[\tilde{\mathbf{R}}(s)]^{-1}\mathbf{D}(s)]^{-1} = \mathbf{G_N}(s)[\mathbf{G_D}(s)]^{-1} \tag{B.6.7}$$

where

$$\mathbf{G_N}(s) \stackrel{\triangle}{=} \tilde{\mathbf{L}}(s)\mathbf{N}(s); \qquad \mathbf{G_D}(s) \stackrel{\triangle}{=} [\tilde{\mathbf{R}}(s)]^{-1}\mathbf{D}(s) \tag{B.6.8}$$

Equations (B.6.7) and (B.6.8) define what is known as a *right matrix fraction description* (RMFD).

It can be shown that $\mathbf{G_D}(s)$ is always column-equivalent to a column proper matrix $\mathbf{P}(s)$. (See definition B.9.) This implies that the degree of the pole polynomial $p_p(s)$ is equal to the sum of the degrees of the columns of $\mathbf{P}(s)$.

We also observe that the RMFD is not unique, because, for any nonsingular $m \times m$ matrix $\mathbf{\Omega}(s)$, we can write $\mathbf{G}(s)$ as

$$\mathbf{G}(s) = \mathbf{G_N}(s)\mathbf{\Omega}(s)[\mathbf{G_D}(s)\mathbf{\Omega}(s)]^{-1} \tag{B.6.9}$$

where $\mathbf{\Omega}(s)$ is said to be a right common factor. When the only right common factors of $\mathbf{G_N}(s)$ and $\mathbf{G_D}(s)$ are unimodular matrices, then, from definition B.7, we have that $\mathbf{G_N}(s)$ and $\mathbf{G_D}(s)$ are right coprime. In this case, we say that the RMFD $(\mathbf{G_N}(s), \mathbf{G_D}(s))$ is **irreducible**.

It is easy to see that when a RMFD is irreducible, then

- $s = z$ is a zero of $\mathbf{G}(s)$ if and only if $\mathbf{G_N}(s)$ loses rank at $s = z$; and
- $s = p$ is a pole of $\mathbf{G}(s)$ if and only if $\mathbf{G_D}(s)$ is singular at $s = p$. This means that the pole polynomial of $\mathbf{G}(s)$ is $p_p(s) = \det(\mathbf{G_D}(s))$.

Remark B.1. *A* left matrix fraction description *(LMFD) can be built similarly, with a different grouping of the matrices in (B.6.7). Namely,*

$$\mathbf{G}(s) = \tilde{\mathbf{L}}(s)[\mathbf{D}(s)]^{-1}\mathbf{N}(s)\tilde{\mathbf{R}}(s) = [\mathbf{D}(s)[\tilde{\mathbf{L}}(s)]^{-1}]^{-1}[\mathbf{N}(s)\tilde{\mathbf{R}}(s)] = [\overline{\mathbf{G}}_\mathbf{D}(s)]^{-1}\overline{\mathbf{G}}_\mathbf{N}(s) \tag{B.6.10}$$

where

$$\overline{\mathbf{G}}_\mathbf{N}(s) \stackrel{\triangle}{=} \mathbf{N}(s)\tilde{\mathbf{R}}(s); \qquad \overline{\mathbf{G}}_\mathbf{D}(s) \stackrel{\triangle}{=} \mathbf{D}(s)[\tilde{\mathbf{L}}(s)]^{-1} \tag{B.6.11}$$

□□□

The left and right matrix descriptions have been initially derived starting from the Smith–McMillan form. Hence, the factors are polynomial matrices. However, it is immediate to see that they provide a more general description. In particular, $\mathbf{G_N}(s)$, $\mathbf{G_D}(s)$, $\overline{\mathbf{G}}_\mathbf{N}(s)$ and $\overline{\mathbf{G}}_\mathbf{N}(s)$ are generally matrices with rational entries. One possible way to obtain this type of representation is to divide the two polynomial matrices forming the original MFD by the same (stable) polynomial.

An example summarizing the above concepts is considered next.

Example B.2. *Consider a 2×2 MIMO system having the transfer function*

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{4}{(s+1)(s+2)} & \dfrac{-0.5}{s+1} \\ \\ \dfrac{1}{s+2} & \dfrac{2}{(s+1)(s+2)} \end{bmatrix} \tag{B.6.12}$$

B.2.1 *Find the Smith–McMillan form by performing elementary row and column operations.*

B.2.2 *Find the poles and zeros.*

B.2.3 *Build a RMFD for the model.*

Solution

B.2.1 *We first compute its Smith–McMillan form by performing elementary row and column operations. Referring to equation* (B.6.1)*, we have that*

$$\mathbf{G^{SM}}(s) = \mathbf{L}(s)\mathbf{G}(s)\mathbf{R}(s) = \begin{bmatrix} \dfrac{1}{(s+1)(s+2)} & 0 \\ 0 & \dfrac{s^2+3s+18}{(s+1)(s+2)} \end{bmatrix} \tag{B.6.13}$$

with

$$\mathbf{L}(s) = \begin{bmatrix} \dfrac{1}{4} & 0 \\ -2(s+1) & 8 \end{bmatrix}; \qquad \mathbf{R}(s) = \begin{bmatrix} 1 & \dfrac{s+2}{8} \\ 0 & 1 \end{bmatrix} \tag{B.6.14}$$

B.2.2 *We see that the observable and controllable part of the system has zero and pole polynomials given by*

$$p_z(s) = s^2 + 3s + 18; \qquad p_p(s) = (s+1)^2(s+2)^2 \tag{B.6.15}$$

which, in turn, implies that there are two transmission zeros, located at $-1.5 \pm j3.97$*, and four poles, located at* $-1, -1, -2$ *and* -2*.*

B.2.3 *We can now build a RMFD by using (B.6.2). We first notice that*

$$\tilde{\mathbf{L}}(s) = [\mathbf{L}(s)]^{-1} = \begin{bmatrix} 4 & 0 \\ s+1 & \frac{1}{8} \end{bmatrix}; \qquad \tilde{\mathbf{R}}(s) = [\mathbf{R}(s)]^{-1} = \begin{bmatrix} 1 & -\frac{s+2}{8} \\ 0 & 0 \end{bmatrix} \tag{B.6.16}$$

Then, using (B.6.6), with

$$\mathbf{N}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s^2+3s+18 \end{bmatrix}; \qquad \mathbf{D}(s) = \begin{bmatrix} (s+1)(s+2) & 0 \\ 0 & (s+1)(s+2) \end{bmatrix} \tag{B.6.17}$$

the RMFD is obtained from (B.6.7), (B.6.16), and (B.6.17), leading to

$$\mathbf{G_N}(s) = \begin{bmatrix} 4 & 0 \\ s+1 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & s^2+3s+18 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ s+1 & \frac{s^2+3s+18}{8} \end{bmatrix} \tag{B.6.18}$$

and

$$\mathbf{G_D}(s) = \begin{bmatrix} 1 & \frac{s+2}{8} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (s+1)(s+2) & 0 \\ 0 & (s+1)(s+2) \end{bmatrix} \tag{B.6.19}$$

$$= \begin{bmatrix} (s+1)(s+2) & \frac{(s+1)(s+2)^2}{8} \\ 0 & (s+1)(s+2) \end{bmatrix} \tag{B.6.20}$$

These can then be turned into proper transfer-function matrices by introducing common stable denominators.

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