Appendix A

# NOTATION, SYMBOLS, AND ACRONYMS

| Notation | Meaning |
|---|---|
| $t$ | Continuous-time variable |
| $f(t)$ | Continuous-time signal |
| $k$ | Discrete-time variable |
| $\{f[k]\}$ | Discrete-time sequence |
| $\Delta$ | Sampling period |
| $f(k\Delta)$ | Sampled version of $f(t)$ |
| $\delta$ | Delta operator |
| $q$ | forward shift operator |
| $\delta_K(k)$ | Kronecker delta |
| $\delta(t)$ | Dirac delta |
| $\mathcal{E}\{...\}$ | Expected value of ... |
| $\Gamma_c$ | Controllability matrix in state space description |
| $\Gamma_o$ | Observability matrix in state space description |
| $\Lambda\{...\}$ | Set of eigenvalues of matrix ... |
| $\mu(t-t_o)$ | unit step (continuous time) at time $t = to$ |
| $\mu[k-k_o]$ | unit step (discrete time) at time $k = k_o$ |
| $f^s(t)$ | Dirac impulse-sampled version of $f(t)$ |
| $\mathcal{F}[...]$ | Fourier transform of ... |
| $\mathcal{L}[...]$ | Laplace transform of ... |
| $\mathcal{D}[...]$ | Delta-transform of ... |
| $\mathcal{Z}[...]$ | Z-transform of ... |
| $\mathcal{F}^{-1}[...]$ | inverse Fourier transform of ... |
| $\mathcal{L}^{-1}[...]$ | inverse Laplace transform of ... |
| $\mathcal{D}^{-1}[...]$ | inverse Delta-transform of ... |

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| Notation | Meaning |
|---|---|
| $\mathcal{Z}^{-1}[...]$ | inverse Z-transform of ... |
| $s$ | Laplace-transform complex variable |
| $\omega$ | angular frequency |
| $\gamma$ | Delta-transform complex variable |
| $z$ | Z-transform complex variable |
| $F(j\omega)$ | Fourier transform of $f(t)$ |
| $F(s)$ | Laplace transform of $f(t)$ |
| $F_\delta(\gamma)$ | Delta-transform of $\{f[k]\}$ |
| $F_q(z)$ | Z-transform of $\{f[k]\}$ |
| $f_1(t) * f_2(t)$ | Time convolution of $f_1(t)$ and $f_2(t)$ |
| $F_1(s) * F_2(s)$ | Complex convolution of $F_1(s)$ and $F_2(s)$ |
| $\Re\{...\}$ | real part of ... |
| $\Im\{...\}$ | imaginary part of ... |
| $\mathbb{C}^{m\times n}$ | set of all $m \times n$ matrices with complex entries |
| $\mathcal{H}_2$ | Hilbert space of those functions square-integrable along the imaginary axis and analytic in the right-half plane |
| $\mathcal{L}_1$ | Hilbert space of those functions absolutely integrable along the imaginary axis |
| $\mathcal{L}_2$ | Hilbert space of those functions square-integrable along the imaginary axis. |
| $\mathcal{H}_\infty$ | Hilbert space of those functions bounded along the imaginary axis and analytic in the right-half plane |
| $\mathcal{RH}_\infty$ | Hilbert space of those **rational** functions bounded along the imaginary axis and analytic in the right-half plane |
| $\mathcal{L}_\infty$ | Hilbert space of those functions bounded along the imaginary axis. |
| $\mathbb{N}$ | set of all natural numbers |
| $\mathbb{R}^+$ | set of real numbers larger than zero |
| $\mathbb{R}^-$ | set of real numbers smaller than zero |
| $\mathbb{R}^{m\times n}$ | set of all $m \times n$ matrices with real entries |
| $\boldsymbol{\mathcal{S}}$ | set of all real rational functions with (finite) poles strictly inside the LHP |
| $\mathbb{Z}$ | set of all integer numbers |
| $[\alpha_{ik}]$ | Matrix where the element in the $i^{th}$ row and $k^{th}$ column is denoted by $\alpha_{ik}$ |
| $[\mathbf{A}]_{ik}$ | element in the $i^{th}$ row and $k^{th}$ of matrix $\mathbf{A}$ |
| $[\mathbf{A}]_{i*}$ | $i^{th}$ row of matrix $\mathbf{A}$ |
| $[\mathbf{A}]_{*k}$ | $k^{th}$ columm of matrix $\mathbf{A}$ |

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| *continued from previous page* | |
|---|---|
| Notation | Meaning |
| $(...)^*$ | complex conjugate of ... |
| $G_{h0}(s)$ | transfer function of a zero-order hold |
| $G_{h1}(s)$ | transfer function of a first-order hold |
| $H\langle ... \rangle$ | operator notation, i.e. $H$ operates on ... |
| $H_1 \otimes H_2\langle ... \rangle$ | composite operators, i.e., $H_1\langle H_2\langle ... \rangle\rangle$ |
| $\mathbf{I_k}$ | identity matrix in $\mathbb{R}^{k\times k}$ |
| d.c. | *direct current*, i.e., zero-frequency signal |
| d.o.f. | degrees of freedom |
| CTARE | Continuous-Time Algebraic Riccati Equation |
| DTARE | Discrete-Time Algebraic Riccati Equation |
| CTDRE | Continuous-Time Dynamic Riccati Equation |
| DTDRE | Discrete-Time Dynamic Riccati Equation |
| IMC | Internal Model Control |
| IMP | Internal Model Principle |
| LHP | left half-plane |
| OLHP | open left half-plane |
| RHP | right half- plane |
| ORHP | open right-half plane |
| NMP | nonminimum phase |
| MFD | Matrix fraction description |
| LMFD | Left matrix fraction description |
| RMFD | Right matrix fraction description |
| LTI | Linear time invariant |
| LQR | Linear quadratic regulator |
| w.r.t | with respect to ... |

**Table A.1.** Notation, symbols and acronyms

# Appendix B

# SMITH–MCMILLAN FORMS

## B.1 Introduction

Smith–McMillan forms correspond to the underlying structures of natural MIMO transfer-function matrices. The key ideas are summarized below.

## B.2 Polynomial Matrices

Multivariable transfer functions depend on polynomial matrices. There are a number of related terms that are used. Some of these are introduced here:

**Definition B.1.** *A matrix* $\mathbf{\Pi}(s) = [p_{ik}(s)] \in \mathbb{R}^{n_1 \times n_2}$ *is a* ***polynomial matrix*** *if* $p_{ik}(s)$ *is a polynomial in* $s$*, for* $i = 1, 2, \ldots, n_1$ *and* $k = 1, 2, \ldots, n_2$.

**Definition B.2.** *A polynomial matrix* $\mathbf{\Pi}(s)$ *is said to be a* ***unimodular matrix*** *if its determinant is a constant. Clearly, the inverse of a unimodular matrix is also a unimodular matrix.*

**Definition B.3.** *An* ***elementary operation*** *on a polynomial matrix is one of the following three operations:*

*(eo1) interchange of two rows or two columns;*

*(eo2) multiplication of one row or one column by a constant;*

*(eo3) addition of one row (column) to another row (column) times a polynomial.*

**Definition B.4.** *A left (right)* ***elementary matrix*** *is a matrix such that, when it multiplies from the left (right) a polynomial matrix, then it performs a row (column) elementary operation on the polynomial matrix. All elementary matrices are unimodular.*

**Definition B.5.** *Two polynomial matrices* $\mathbf{\Pi}_1(s)$ *and* $\mathbf{\Pi}_2(s)$ *are* ***equivalent matrices****, if there exist sets of left and right elementary matrices,* $\{\mathbf{L}_1(s), \mathbf{L}_2(s), \ldots, \mathbf{L}_{k1}\}$ *and* $\{\mathbf{R}_1(s), \mathbf{R}_2(s), \ldots, \mathbf{R}_{k2}\}$*, respectively, such that*

$$\mathbf{\Pi}_1(s) = \mathbf{L}_{k1}(s) \cdots \mathbf{L}_2(s)\mathbf{L}_1\mathbf{\Pi}_2(s)\mathbf{R}_1(s)\mathbf{R}_2(s) \cdots \mathbf{R}_{k2} \qquad \text{(B.2.1)}$$

**Definition B.6.** *The* ***rank of a polynomial matrix*** *is the rank of the matrix almost everywhere in $s$. The definition implies that the rank of a polynomial matrix is independent of the argument.*

**Definition B.7.** *Two polynomial matrices $\mathbf{V}(s)$ and $\mathbf{W}(s)$ having the same number of columns (rows) are* ***right (left) coprime*** *if all common right (left) factors are unimodular matrices.*

**Definition B.8.** *The degree $\partial_{ck}$ ($\partial_{rk}$) of the $k^{th}$ column (row) $[\mathbf{V}(s)]_{*k}$ ( $[\mathbf{V}(s)]_{k*}$) of a polynomial matrix $\mathbf{V}(s)$ is the degree of highest power of $s$ in that column (row).*

**Definition B.9.** *A polynomial matrix $\mathbf{V}(s) \in \mathbb{C}^{m\times m}$ is column proper if*

$$\lim_{s\to\infty} \det(\mathbf{V}(s)\,\mathrm{diag}\left(s^{-\partial_{c1}}, s^{-\partial_{c2}}, \ldots, s^{-\partial_{cm}}\right)) \tag{B.2.2}$$

*has a finite, nonzero value.*

**Definition B.10.** *A polynomial matrix $\mathbf{V}(s) \in \mathbb{C}^{m\times m}$ is row proper if*

$$\lim_{s\to\infty} \det(\mathrm{diag}\left(s^{-\partial_{r1}}, s^{-\partial_{r2}}, \ldots, s^{-\partial_{rm}}\right)\mathbf{V}(s)) \tag{B.2.3}$$

*has a finite, nonzero value.*

## B.3 Smith Form for Polynomial Matrices

Using the above notation, we can manipulate polynomial matrices in ways that mirror the ways we manipulate matrices of reals. For example, the following result describes a diagonal form for polynomial matrices.

**Theorem B.1 (Smith form).** *Let $\mathbf{\Pi}(s)$ be a $m_1 \times m_2$ polynomial matrix of rank $r$; then $\mathbf{\Pi}(s)$ is equivalent to either a matrix $\mathbf{\Pi}_f(s)$ (for $m_1 < m_2$) or to a matrix $\mathbf{\Pi}_c(s)$ (for $m_2 < m_1$), with*

$$\mathbf{\Pi}_f(s) = \begin{bmatrix}\mathbf{E}(s) & \Theta_f\end{bmatrix}; \qquad \mathbf{\Pi}_c(s) = \begin{bmatrix}\mathbf{E}(s)\\ \Theta_c\end{bmatrix} \tag{B.3.1}$$

$$\mathbf{E}(s) = \mathrm{diag}(\overline{\epsilon}_1(s), \ldots, \overline{\epsilon}_r(s), 0, \ldots, 0) \tag{B.3.2}$$

*where $\Theta_f$ and $\Theta_c$ are matrices with all their elements equal to zero.*

*Furthermore $\overline{\epsilon}_i(s)$ are monic polynomials for $i = 1, 2, \ldots, r$, such that $\overline{\epsilon}_i(s)$ is a factor in $\overline{\epsilon}_{i+1}(s)$, i.e. $\overline{\epsilon}_i(s)$ divides $\overline{\epsilon}_{i+1}(s)$.*

*If $m_1 = m_2$, then $\mathbf{\Pi}(s)$ is equivalent to the square matrix $\mathbf{E}(s)$.*

### Proof (by construction)

*(i) By performing row and column interchange operations on $\mathbf{\Pi}(s)$, bring to position (1,1) the least degree polynomial entry in $\mathbf{\Pi}(s)$. Say this minimum degree is $\nu_1$*

*(ii)* *Using elementary operation (e03) (see definition B.3), reduce the term in the position (2,1) to degree* $\nu_2 < \nu_1$. *If the term in position (2,1) becomes zero, then go to the next step, otherwise, interchange rows 1 and 2 and repeat the procedure until the term in position (2,1) becomes zero.*

*(iii)* *Repeat step (ii) with the other elements in the first column.*

*(iv)* *Apply the same procedure to all the elements but the first one in the first row.*

*(v)* *Go back to step (ii) if nonzero entries due to step (iv) appear in the first column. Notice that the degree of the entry (1,1) will fall in each cycle, until we finally end up with a matrix which can be partitioned as*

$$\mathbf{\Pi}(s) = \begin{bmatrix} \overline{\pi}_{11}^{(j)}(s) & 0 \quad 0 \quad \dots \quad 0 \quad 0 \\ 0 & \\ 0 & \\ \vdots & \mathbf{\Pi_j}(s) \\ 0 & \\ 0 & \end{bmatrix} \tag{B.3.3}$$

*where* $\overline{\pi}_{11}^{(j)}(s)$ *is a monic polynomial.*

*(vi)* *If there is an element of* $\mathbf{\Pi_j}(s)$ *which is of lesser degree than* $\overline{\pi}_{11}^{(j)}(s)$, *then add the column where this element is to the first column and repeat steps (ii) to (v). Do this until the form* (B.3.3) *is achieved with* $\overline{\pi}_{11}^{(j)}(s)$ *of less or, at most, equal degree to that of every element in* $\mathbf{\Pi_j}(s)$. *This will yield further reduction in the degree of the entry in position (1,1).*

*(vii)* *Make* $\overline{\epsilon}_1(s) = \overline{\pi}_{11}^{(j)}(s)$.

*(viii)* *Repeat the procedure from steps (i) through (viii) to matrix* $\mathbf{\Pi_j}(s)$.

Actually the polynomials $\overline{\epsilon}_i(s)$ in the above result can be obtained in a direct fashion, as follows:

(i) Compute all minor determinants of $\mathbf{\Pi}(s)$.

(ii) Define $\chi_i(s)$ as the (monic) greatest common divisor (g.c.d.) of all $i \times i$ minor determinants of $\mathbf{\Pi}(s)$. Make $\chi_0(s) = 1$.

(iii) Compute the polynomials $\overline{\epsilon}_i(s)$ as

$$\overline{\epsilon}_i(s) = \frac{\chi_i(s)}{\chi_{i-1}(s)} \tag{B.3.4}$$

## B.4 Smith–McMillan Form for Rational Matrices

A straightforward application of Theorem B.1 leads to the following result, which gives a diagonal form for a rational transfer-function matrix:

**Theorem B.2 (Smith–McMillan form).** *Let $\mathbf{G}(s) = [G_{ik}(s)]$ be an $m \times m$ matrix transfer function, where $G_{ik}(s)$ are rational scalar transfer functions:*

$$\mathbf{G}(s) = \frac{\mathbf{\Pi}(s)}{D_G(s)} \tag{B.4.1}$$

*where $\mathbf{\Pi}(s)$ is an $m \times m$ polynomial matrix of rank $r$ and $D_G(s)$ is the least common multiple of the denominators of all elements $G_{ik}(s)$.*

*Then, $\mathbf{G}(s)$ is equivalent to a matrix $\mathbf{M}(s)$, with*

$$\mathbf{M}(s) = \text{diag}\left(\frac{\epsilon_1(s)}{\delta_1(s)}, \ldots, \frac{\epsilon_r(s)}{\delta_r(s)}, 0, \ldots, 0\right) \tag{B.4.2}$$

*where $\{\epsilon_i(s), \delta_i(s)\}$ is a pair of monic and coprime polynomials for $i = 1, 2, \ldots, r$.*

*Furthermore, $\epsilon_i(s)$ is a factor of $\epsilon_{i+1}(s)$ and $\delta_i(s)$ is a factor of $\delta_{i-1}(s)$.*

### Proof

*We write the transfer-function matrix as in (B.4.1). We then perform the algorithm outlined in Theorem B.1 to convert $\mathbf{\Pi}(s)$ to Smith normal form. Finally, canceling terms for the denominator $D_G(s)$ leads to the form given in (B.4.2).*

□□□

We use the symbol $\mathbf{G}^{SM}(s)$ to denote $\mathbf{M}(s)$, which is the *Smith–McMillan form* of the transfer-function matrix $\mathbf{G}(s)$ .

We illustrate the formula of the Smith–McMillan form by a simple example.

**Example B.1.** *Consider the following transfer-function matrix*

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{4}{(s+1)(s+2)} & \dfrac{-1}{s+1} \\ \\ \dfrac{2}{s+1} & \dfrac{-1}{2(s+1)(s+2)} \end{bmatrix} \tag{B.4.3}$$

*We can then express $\mathbf{G}(s)$ in the form (B.4.1):*

$$\mathbf{G}(s) = \frac{\mathbf{\Pi}(s)}{D_G(s)}; \quad \mathbf{\Pi}(s) = \begin{bmatrix} 4 & -(s+2) \\ 2(s+2) & -\dfrac{1}{2} \end{bmatrix}; \quad D_G(s) = (s+1)(s+2) \tag{B.4.4}$$

*The polynomial matrix $\mathbf{\Pi}(s)$ can be reduced to the Smith form defined in Theorem B.1. To do that, we first compute its greatest common divisors:*

$$\chi_0 = 1 \tag{B.4.5}$$

$$\chi_1 = gcd\left\{4; -(s+2); 2(s+2); -\frac{1}{2}\right\} = 1 \tag{B.4.6}$$

$$\chi_2 = gcd\{2s^2+8s+6\} = s^2+4s+3 = (s+1)(s+3) \tag{B.4.7}$$

*This leads to*

$$\bar{\epsilon}_1 = \frac{\chi_1}{\chi_0} = 1; \qquad \bar{\epsilon}_2 = \frac{\chi_2}{\chi_1} = (s+1)(s+3) \tag{B.4.8}$$

*From here, the Smith–McMillan form can be computed to yield*

$$\mathbf{G^{SM}}(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & 0 \\ 0 & \frac{s+3}{s+2} \end{bmatrix} \tag{B.4.9}$$

## B.5 Poles and Zeros

The Smith–McMillan form can be utilized to give an unequivocal definition of poles and zeros in the multivariable case. In particular, we have:

**Definition B.11.** *Consider a transfer-function matrix , $\mathbf{G}(s)$.*

**(i)** *$p_z(s)$ and $p_p(s)$ are said to be the **zero polynomial** and the **pole polynomial** of $\mathbf{G}(s)$, respectively, where*

$$p_z(s) \stackrel{\triangle}{=} \epsilon_1(s)\epsilon_2(s)\cdots\epsilon_r(s); \qquad p_p(s) \stackrel{\triangle}{=} \delta_1(s)\delta_2(s)\cdots\delta_r(s) \tag{B.5.1}$$

*and where $\epsilon_1(s)$, $\epsilon_2(s)$, ..., $\epsilon_r(s)$ and $\delta_1(s)$, $\delta_2(s)$, ..., $\delta_r(s)$ are the polynomials in the Smith–McMillan form, $\mathbf{G}^{SM}(s)$ of $\mathbf{G}(s)$.*

*Note that $p_z(s)$ and $p_p(s)$ are monic polynomials.*

**(ii)** *The **zeros** of the matrix $\mathbf{G}(s)$ are defined to be the roots of $p_z(s)$, and the **poles** of $\mathbf{G}(s)$ are defined to be the roots of $p_p(s)$.*

**(iii)** *The **McMillan degree** of $\mathbf{G}(s)$ is defined as the degree of $p_p(s)$.*

In the case of square plants (same number of inputs as outputs), it follows that $\det[\mathbf{G}(s)]$ is a simple function of $p_z(s)$ and $p_p(s)$. Specifically, we have

$$\det[\mathbf{G}(s)] = K_\infty \frac{p_z(s)}{p_p(s)} \tag{B.5.2}$$

Note, however, that $p_z(s)$ and $p_p(s)$ are not necessarily coprime. Hence, the scalar rational function $\det[\mathbf{G}(s)]$ is not sufficient to determine all zeros and poles of $\mathbf{G}(s)$. However, the relative degree of $\det[\mathbf{G}(s)]$ is equal to the difference between the number of poles and the number of zeros of the MIMO transfer-function matrix.

## B.6 Matrix Fraction Descriptions (MFD)

A model structure that is related to the Smith–McMillan form is that of a **matrix fraction description** (MFD). There are two types, namely a right matrix fraction description (RMFD) and a left matrix fraction description (LMFD).

We recall that a matrix $\mathbf{G}(s)$ and its Smith-McMillan form $\mathbf{G^{SM}}(s)$ are equivalent matrices. Thus, there exist two unimodular matrices, $\mathbf{L}(s)$ and $\mathbf{R}(s)$, such that

$$\mathbf{G^{SM}}(s) = \mathbf{L}(s)\mathbf{G}(s)\mathbf{R}(s) \tag{B.6.1}$$

This implies that if $\mathbf{G}(s)$ is an $m \times m$ proper transfer-function matrix, then there exist a $m \times m$ matrix $\tilde{\mathbf{L}}(s)$ and an $m \times m$ matrix $\tilde{\mathbf{R}}(s)$, such as

$$\mathbf{G}(s) = \tilde{\mathbf{L}}(s)\mathbf{G^{SM}}(s)\tilde{\mathbf{R}}(s) \tag{B.6.2}$$

where $\tilde{\mathbf{L}}(s)$ and $\tilde{\mathbf{R}}(s)$ are, for example, given by

$$\tilde{\mathbf{L}}(s) = [\mathbf{L}(s)]^{-1}; \qquad \tilde{\mathbf{R}}(s) = [\mathbf{R}(s)]^{-1} \tag{B.6.3}$$

We next define the following two matrices:

$$\mathbf{N}(s) \stackrel{\triangle}{=} \operatorname{diag}(\epsilon_1(s), \ldots, \epsilon_r(s), 0, \ldots, 0) \tag{B.6.4}$$

$$\mathbf{D}(s) \stackrel{\triangle}{=} \operatorname{diag}(\delta_1(s), \ldots, \delta_r(s), 1, \ldots, 1) \tag{B.6.5}$$

where $\mathbf{N}(s)$ and $\mathbf{D}(s)$ are $m \times m$ matrices. Hence, $\mathbf{G^{SM}}(s)$ can be written as

$$\mathbf{G^{SM}}(s) = \mathbf{N}(s)[\mathbf{D}(s)]^{-1} \tag{B.6.6}$$

Combining (B.6.2) and (B.6.6), we can write

$$\mathbf{G}(s) = \tilde{\mathbf{L}}(s)\mathbf{N}(s)[\mathbf{D}(s)]^{-1}\tilde{\mathbf{R}}(s) = [\tilde{\mathbf{L}}(s)\mathbf{N}(s)][[\tilde{\mathbf{R}}(s)]^{-1}\mathbf{D}(s)]^{-1} = \mathbf{G_N}(s)[\mathbf{G_D}(s)]^{-1} \tag{B.6.7}$$

where

$$\mathbf{G_N}(s) \stackrel{\triangle}{=} \tilde{\mathbf{L}}(s)\mathbf{N}(s); \qquad \mathbf{G_D}(s) \stackrel{\triangle}{=} [\tilde{\mathbf{R}}(s)]^{-1}\mathbf{D}(s) \tag{B.6.8}$$

Equations (B.6.7) and (B.6.8) define what is known as a *right matrix fraction description* (RMFD).

It can be shown that $\mathbf{G_D}(s)$ is always column-equivalent to a column proper matrix $\mathbf{P}(s)$. (See definition B.9.) This implies that the degree of the pole polynomial $p_p(s)$ is equal to the sum of the degrees of the columns of $\mathbf{P}(s)$.

We also observe that the RMFD is not unique, because, for any nonsingular $m \times m$ matrix $\mathbf{\Omega}(s)$, we can write $\mathbf{G}(s)$ as

$$\mathbf{G}(s) = \mathbf{G_N}(s)\mathbf{\Omega}(s)[\mathbf{G_D}(s)\mathbf{\Omega}(s)]^{-1} \tag{B.6.9}$$

where $\mathbf{\Omega}(s)$ is said to be a right common factor. When the only right common factors of $\mathbf{G_N}(s)$ and $\mathbf{G_D}(s)$ are unimodular matrices, then, from definition B.7, we have that $\mathbf{G_N}(s)$ and $\mathbf{G_D}(s)$ are right coprime. In this case, we say that the RMFD $(\mathbf{G_N}(s), \mathbf{G_D}(s))$ is **irreducible**.

It is easy to see that when a RMFD is irreducible, then

- $s = z$ is a zero of $\mathbf{G}(s)$ if and only if $\mathbf{G_N}(s)$ loses rank at $s = z$; and
- $s = p$ is a pole of $\mathbf{G}(s)$ if and only if $\mathbf{G_D}(s)$ is singular at $s = p$. This means that the pole polynomial of $\mathbf{G}(s)$ is $p_p(s) = \det(\mathbf{G_D}(s))$.

**Remark B.1.** *A* left matrix fraction description *(LMFD) can be built similarly, with a different grouping of the matrices in (B.6.7). Namely,*

$$\mathbf{G}(s) = \tilde{\mathbf{L}}(s)[\mathbf{D}(s)]^{-1}\mathbf{N}(s)\tilde{\mathbf{R}}(s) = [\mathbf{D}(s)[\tilde{\mathbf{L}}(s)]^{-1}]^{-1}[\mathbf{N}(s)\tilde{\mathbf{R}}(s)] = [\overline{\mathbf{G}}_\mathbf{D}(s)]^{-1}\overline{\mathbf{G}}_\mathbf{N}(s) \tag{B.6.10}$$

*where*

$$\overline{\mathbf{G}}_\mathbf{N}(s) \stackrel{\triangle}{=} \mathbf{N}(s)\tilde{\mathbf{R}}(s); \qquad \overline{\mathbf{G}}_\mathbf{D}(s) \stackrel{\triangle}{=} \mathbf{D}(s)[\tilde{\mathbf{L}}(s)]^{-1} \tag{B.6.11}$$

□□□

The left and right matrix descriptions have been initially derived starting from the Smith–McMillan form. Hence, the factors are polynomial matrices. However, it is immediate to see that they provide a more general description. In particular, $\mathbf{G_N}(s)$, $\mathbf{G_D}(s)$, $\overline{\mathbf{G}}_\mathbf{N}(s)$ and $\overline{\mathbf{G}}_\mathbf{N}(s)$ are generally matrices with rational entries. One possible way to obtain this type of representation is to divide the two polynomial matrices forming the original MFD by the same (stable) polynomial.

An example summarizing the above concepts is considered next.

**Example B.2.** *Consider a $2 \times 2$ MIMO system having the transfer function*

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{4}{(s+1)(s+2)} & \dfrac{-0.5}{s+1} \\ \\ \dfrac{1}{s+2} & \dfrac{2}{(s+1)(s+2)} \end{bmatrix} \tag{B.6.12}$$

**B.2.1** *Find the Smith–McMillan form by performing elementary row and column operations.*

**B.2.2** *Find the poles and zeros.*

**B.2.3** *Build a RMFD for the model.*

## Solution

**B.2.1** *We first compute its Smith–McMillan form by performing elementary row and column operations. Referring to equation* (B.6.1)*, we have that*

$$\mathbf{G^{SM}}(s) = \mathbf{L}(s)\mathbf{G}(s)\mathbf{R}(s) = \begin{bmatrix} \dfrac{1}{(s+1)(s+2)} & 0 \\ 0 & \dfrac{s^2+3s+18}{(s+1)(s+2)} \end{bmatrix} \tag{B.6.13}$$

*with*

$$\mathbf{L}(s) = \begin{bmatrix} \dfrac{1}{4} & 0 \\ -2(s+1) & 8 \end{bmatrix}; \qquad \mathbf{R}(s) = \begin{bmatrix} 1 & \dfrac{s+2}{8} \\ 0 & 1 \end{bmatrix} \tag{B.6.14}$$

**B.2.2** *We see that the observable and controllable part of the system has zero and pole polynomials given by*

$$p_z(s) = s^2 + 3s + 18; \qquad p_p(s) = (s+1)^2(s+2)^2 \tag{B.6.15}$$

*which, in turn, implies that there are two transmission zeros, located at* $-1.5 \pm j3.97$*, and four poles, located at* $-1, -1, -2$ *and* $-2$*.*

**B.2.3** *We can now build a RMFD by using (B.6.2). We first notice that*

$$\tilde{\mathbf{L}}(s) = [\mathbf{L}(s)]^{-1} = \begin{bmatrix} 4 & 0 \\ s+1 & \frac{1}{8} \end{bmatrix}; \qquad \tilde{\mathbf{R}}(s) = [\mathbf{R}(s)]^{-1} = \begin{bmatrix} 1 & -\frac{s+2}{8} \\ 0 & 0 \end{bmatrix} \tag{B.6.16}$$

*Then, using (B.6.6), with*

$$\mathbf{N}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s^2+3s+18 \end{bmatrix}; \qquad \mathbf{D}(s) = \begin{bmatrix} (s+1)(s+2) & 0 \\ 0 & (s+1)(s+2) \end{bmatrix} \tag{B.6.17}$$

*the RMFD is obtained from (B.6.7), (B.6.16), and (B.6.17), leading to*

$$\mathbf{G_N}(s) = \begin{bmatrix} 4 & 0 \\ s+1 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & s^2+3s+18 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ s+1 & \frac{s^2+3s+18}{8} \end{bmatrix} \tag{B.6.18}$$

*and*

$$\mathbf{G_D}(s) = \begin{bmatrix} 1 & \frac{s+2}{8} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (s+1)(s+2) & 0 \\ 0 & (s+1)(s+2) \end{bmatrix} \tag{B.6.19}$$

$$= \begin{bmatrix} (s+1)(s+2) & \frac{(s+1)(s+2)^2}{8} \\ 0 & (s+1)(s+2) \end{bmatrix} \tag{B.6.20}$$

*These can then be turned into proper transfer-function matrices by introducing common stable denominators.*

□□□

Appendix C

# RESULTS FROM ANALYTIC FUNCTION THEORY

## C.1 Introduction

This appendix summarizes key results from analytic function theory leading to the Cauchy Integral formula and its consequence, the Poisson–Jensen formula.

## C.2 Independence of Path

Consider functions of two independent variables, $x$ and $y$. (The reader can think of $x$ as the real axis and $y$ as the imaginary axis.)

Let $P(x, y)$ and $Q(x, y)$ be two functions of $x$ and $y$, continuous in some domain $D$. Say we have a curve $C$ in $D$, described by the parametric equations

$$x = f_1(t), \qquad y = f_2(t) \tag{C.2.1}$$

We can then define the following line integrals along the path $C$ from point $A$ to point $B$ inside $D$.

$$\int_A^B P(x, y)dx = \int_{t_1}^{t_2} P(f_1(t), f_2(t))\frac{df_1(t)}{dt}dt \tag{C.2.2}$$

$$\int_A^B Q(x, y)dy = \int_{t_1}^{t_2} Q(f_1(t), f_2(t))\frac{df_2(t)}{dt}dt \tag{C.2.3}$$

**Definition C.1.** *The line integral* $\int Pdx + Qdy$ *is said to be* **independent of the path** *in $D$ if, for every pair of points $A$ and $B$ in $D$, the value of the integral is independent of the path followed from $A$ to $B$.*

□□□

We then have the following result.

**Theorem C.1.** *If $\int Pdx + Qdy$ is independent of the path in $D$, then there exists a function $F(x, y)$ in $D$ such that*

$$\frac{\partial F}{\partial x} = P(x, y); \qquad \frac{\partial F}{\partial y} = Q(x, y) \tag{C.2.4}$$

*hold throughout $D$. Conversely, if a function $F(x, y)$ can be found such that (C.2.4) hold, then $\int Pdx + Qdy$ is independent of the path.*

**Proof**

Suppose that the integral is independent of the path in $D$. Then, choose a point $(x_0, y_0)$ in $D$ and let $F(x, y)$ be defined as follows

$$F(x, y) = \int_{x_0, y_0}^{x, y} Pdx + Qdy \tag{C.2.5}$$

where the integral is taken on an arbitrary path in $D$ joining $(x_0, y_0)$ and $(x, y)$. Because the integral is independent of the path, the integral does indeed depend only on $(x, y)$ and defines the function $F(x, y)$. It remains to establish (C.2.4).

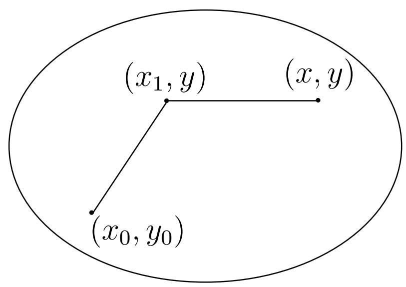


**Figure C.1.** Integration path

For a particular $(x, y)$ in $D$, choose $(x_1, y)$ so that $x_1 \neq x$ and so that the line segment from $(x_1, y)$ to $(x, y)$ in $D$ is as shown in Figure C.1. Because of independence of the path,

$$F(x, y) = \int_{x_0, y_0}^{x_1, y} (Pdx + Qdy) + \int_{x_1, y}^{x, y} (Pdx + Qdy) \tag{C.2.6}$$

We think of $x_1$ and $y$ as being fixed while $(x, y)$ may vary along the horizontal line segment. Thus $F(x, y)$ is being considered as function of $x$. The first integral on the right-hand side of (C.2.6) is then independent of $x$.

Hence, for fixed $y$, we can write

$$F(x,y) = \text{constant} + \int_{x_1}^{x} P(x,y)dx \tag{C.2.7}$$

The fundamental theorem of Calculus now gives

$$\frac{\partial F}{\partial x} = P(x,y) \tag{C.2.8}$$

A similar argument shows that

$$\frac{\partial F}{\partial y} = Q(x,y) \tag{C.2.9}$$

Conversely, let (C.2.4) hold for some $F$. Then, with $t$ as a parameter,

$$\begin{aligned}
F(x,y) = \int_{x_1,y_1}^{x_2,y_2} Pdx + Qdy &= \int_{t_1}^{t_2} \left( \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} \right) dt && \text{(C.2.10)} \\
&= \int_{t_1}^{t_2} \frac{\partial F}{\partial t} dt && \text{(C.2.11)} \\
&= F(x_2,y_2) - F(x_1,y_1) && \text{(C.2.12)}
\end{aligned}$$

□□□

**Theorem C.2.** *If the integral $\int Pdx + Qdy$ is independent of the path in $D$, then*

$$\oint Pdx + Qdy = 0 \tag{C.2.13}$$

*on every closed path in $D$. Conversely if (C.2.13 ) holds for every simple closed path in $D$, then $\int Pdx + Qdy$ is independent of the path in $D$.*

## Proof

Suppose that the integral is independent of the path. Let $C$ be a simple closed path in $D$, and divide $C$ into arcs $\vec{AB}$ and $\vec{BA}$ as in Figure C.2.

$$\begin{aligned}
\oint_C (Pdx + Qdy) &= \int_{AB} Pdx + Qdy + \int_{BA} Pdx + Qdy && \text{(C.2.14)} \\
&= \int_{AB} Pdx + Qdy - \int_{AB} Pdx + Qdy && \text{(C.2.15)}
\end{aligned}$$

The converse result is established by reversing the above argument.

□□□

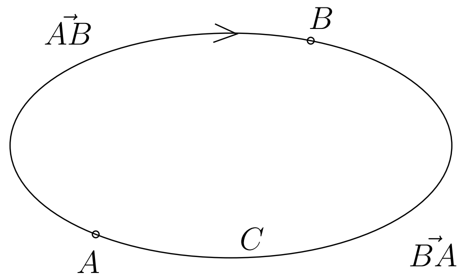


**Figure C.2.** Integration path

**Theorem C.3.** *If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives in $D$ and $\int Pdx + Qdy$ is independent of the path in $D$, then*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \textit{in } D \tag{C.2.16}$$

**Proof**

By Theorem C.1, there exists a function $F$ such that (C.2.4) holds. Equation (C.2.16) follows by partial differentiation.

□□□

Actually, we will be particularly interested in the converse to Theorem C.3. However, this holds under slightly more restrictive assumptions, namely a simply connected domain.

## C.3 Simply Connected Domains

Roughly speaking, a domain $D$ is simply connected if it has no holes. More precisely, $D$ is simply connected if, for every simple closed curve $C$ in $D$, the region $R$ enclosed by $C$ lies wholly in $D$. For simply connected domains we have the following:

**Theorem C.4 (Green's theorem).** *Let $D$ be a simply connected domain, and let $C$ be a piecewise-smooth simple closed curve in $D$. Let $P(x,y)$ and $Q(x,y)$ be functions that are continuous and that have continuous first partial derivatives in $D$. Then*

$$\oint (Pdx + Qdy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \tag{C.3.1}$$

*where $R$ is the region bounded by $C$.*

**Proof**

We first consider a simple case in which $R$ is representable in both of the forms:

$$f_1(x) \leq f_2(x) \qquad \text{for } a \leq x \leq b \tag{C.3.2}$$

$$g_1(y) \leq g_2(y) \qquad \text{for } c \leq y \leq d \tag{C.3.3}$$

Then

$$\int\int_R \frac{\partial P}{\partial y} dxdy = \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y} dxdy \tag{C.3.4}$$

One can now integrate to achieve

$$\int\int_R \frac{\partial P}{\partial y} dxdy = \int_a^b [P(x, f_2(x)) - P(x, f_1(x))]dx \tag{C.3.5}$$

$$= \int_a^b P(x, f_2(x))dx - \int_a^b P(x, f_1(x))dx \tag{C.3.6}$$

$$= \oint_C P(x, y)dx \tag{C.3.7}$$

By a similar argument,

$$\int\int_R \frac{\partial Q}{\partial x} dxdy = \oint_C Q(x, y)dy \tag{C.3.8}$$

For more complex regions, we decompose into simple regions as above. The result then follows.

□□□

We then have the following converse to Theorem C.3.

**Theorem C.5.** *Let $P(x, y)$ and $Q(x, y)$ have continuous derivatives in $D$ and let $D$ be simply connected. If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\oint Pdx + Qdy$ is independent of path in $D$.*

**Proof**

Suppose that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{in } D \tag{C.3.9}$$

Then, by Green's Theorem (Theorem C.4),

$$\oint_c Pdx + Qdy = \int\int_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = 0 \tag{C.3.10}$$

□□□

## C.4 Functions of a Complex Variable

In the sequel, we will let $z = x + jy$ denote a complex variable. Note that $z$ is not the argument in the Z-transform, as used at other points in the book. Also, a function $f(z)$ of a complex variable is equivalent to a function $\bar{f}(x, y)$. This will have real and imaginary parts $u(x, y)$ and $v(x, y)$ respectively.

We can thus write

$$f(z) = u(x, y) + jv(x, y) \tag{C.4.1}$$

Note that we also have

$$\begin{aligned}\int_C f(z)dz &= \int_C (u(x, y) + jv(x, y))(dx + jdy) \\ &= \int_C u(x, y)dx - \int_C v(x, y)dy + j\left\{\int_C u(x, y)dy + \int_C v(x, y)dx\right\}\end{aligned}$$

We then see that the previous results are immediately applicable to the real and imaginary parts of integrals of this type.

## C.5 Derivatives and Differentials

Let $w = f(z)$ be a given complex function of the complex variable $z$. Then $w$ is said to have a derivative at $z_0$ if

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{C.5.1}$$

exists and is independent of the direction of $\Delta z$. We denote this limit, when it exists, by $f'(z_0)$.

## C.6 Analytic Functions

**Definition C.2.** *A function $f(z)$ is said to be analytic in a domain $D$ if $f$ has a continuous derivative in $D$.*

□□□

**Theorem C.6.** *If $w = f(z) = u+jv$ is analytic in $D$, then $u$ and $v$ have continuous partial derivatives satisfying the Cauchy-Riemman conditions.*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{C.6.1}$$

*Furthermore*

$$\frac{\partial w}{\partial z} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + j\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - j\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y} \tag{C.6.2}$$

**Proof**

Let $z_0$ be a fixed point in $D$ and let $\Delta\omega = f(z_0+\Delta z) - f(z_0)$. Because $f$ is analytic, we have

$$\Delta\omega = \gamma\Delta z + \epsilon\Delta z; \qquad \gamma \stackrel{\triangle}{=} f'(z_0) \tag{C.6.3}$$

where $\gamma = a + jb$ and $\epsilon$ goes to zero as $|z_0|$ goes to zero. Then

$$\Delta u + j\Delta v = (a + jb)(\Delta x + j\Delta y) + (\epsilon_1 + j\epsilon_2)(\Delta x + j\Delta y) \tag{C.6.4}$$

So

$$\Delta u = a\Delta x - b\Delta y + \epsilon_1\Delta x - \epsilon_2\Delta y \tag{C.6.5}$$

$$\Delta v = b\Delta x + a\Delta y + \epsilon_2\Delta x + \epsilon_1\Delta y \tag{C.6.6}$$

Thus, in the limit, we can write

$$du = adx - bdy; \qquad dv = bdx - ady \tag{C.6.7}$$

or

$$\frac{\partial u}{\partial x} = a = -\frac{\partial v}{\partial y}; \qquad \frac{\partial u}{\partial y} = -b = -\frac{\partial v}{\partial x} \tag{C.6.8}$$

□□□

Actually, most functions that we will encounter will be analytic, provided the derivative exists. We illustrate this with some examples.

**Example C.1.** *Consider the function $f(z) = z^2$. Then*

$$f(z) = (x + jy)^2 = x^2 - y^2 + j(2xy) = u + jv \tag{C.6.9}$$

*The partial derivatives are*

$$\frac{\partial u}{\partial x} = 2x; \qquad \frac{\partial v}{\partial x} = 2y; \qquad \frac{\partial u}{\partial y} = -2y; \qquad \frac{\partial v}{\partial y} = 2x \tag{C.6.10}$$

*Hence, the function is clearly analytic.*

**Example C.2.** *Consider* $f(z) = |z|$ .

*This function is not analytic, because* $d|z|$ *is a real quantity and, hence,* $\frac{d|z|}{dz}$ *will depend on the direction of* $z$.

**Example C.3.** *Consider a rational function of the form:*

$$W(z) = K\frac{(z-\beta_1)(z-\beta_2)\cdots(z-\beta_m)}{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)} = \frac{N(z)}{D(z)} \tag{C.6.11}$$

$$\frac{\partial W}{\partial z} = \frac{1}{D^2(z)}\left[D(z)\frac{\partial N(z)}{\partial z} - N(z)\frac{\partial D(z)}{\partial z}\right] \tag{C.6.12}$$

These derivatives clearly exist, save when $D = 0$, that is at the poles of $W(z)$.

**Example C.4.** *Consider the same function* $W(z)$ *defined in (C.6.11). Then*

$$\frac{\partial \ln(W)}{\partial z} = \frac{1}{N(z)D(z)}\left[D(z)\frac{\partial N(z)}{\partial z} - N(z)\frac{\partial D(z)}{\partial z}\right] = \frac{1}{N(z)}\frac{\partial N(z)}{\partial z} - \frac{1}{D(z)}\frac{\partial D(z)}{\partial z} \tag{C.6.13}$$

Hence, $\ln(W(z))$ is analytic, save at the poles and zeros of $W(z)$.

## C.7 Integrals Revisited

**Theorem C.7 (Cauchy Integral Theorem).** *If* $f(z)$ *is analytic in some simply connected domain* $D$*, then* $\int f(z)dz$ *is independent of path in* $D$ *and*

$$\oint_C f(z)dz = 0 \tag{C.7.1}$$

*where* $C$ *is a simple closed path in* $D$.

### Proof

This follows from the Cauchy–Riemann conditions together with Theorem C.2.

□□□

We are also interested in the value of integrals in various limiting situations. The following examples cover relevant cases.

We note that if $L_C$ is the length of a simple curve $C$, then

$$\left| \int_C f(z)dz \right| \leq \max_{z \in C}(|f(z)|)L_C \tag{C.7.2}$$

**Example C.5.** *Assume that $C$ is a semicircle centered at the origin and having radius $R$. The path length is then $L_C = \pi R$. Hence,*

- *if $f(z)$ varies as $z^{-2}$, then $|f(z)|$ on $C$ must vary as $R^{-2}$ – hence, the integral on $C$ vanishes for $R \to \infty$.*
- *if $f(z)$ varies as $z^{-1}$, then $|f(z)|$ on $C$ must vary as $R^{-1}$ – then, the integral on $C$ becomes a constant as $R \to \infty$.*

**Example C.6.** *Consider the function $f(z) = \ln(z)$ and an arc of a circle, $C$, described by $z = \epsilon e^{j\gamma}$ for $\gamma \in [-\gamma_1, \gamma_1]$. Then*

$$I_\epsilon \triangleq \lim_{\epsilon \to 0} \int_C f(z)dz = 0 \tag{C.7.3}$$

*This is proven as follows. On $C$, we have that $f(z) = \ln(\epsilon)$. Then*

$$I_\epsilon = \lim_{\epsilon \to 0} [(\gamma_2 - \gamma_1)\epsilon \ln(\epsilon)] \tag{C.7.4}$$

*We then use the fact that $\lim_{|x| \to 0}(x \ln x) = 0$, and the result follows.*

**Example C.7.** *Consider the function*

$$f(z) = \ln\left(1 + \frac{a}{z^n}\right) \qquad n \geq 1 \tag{C.7.5}$$

*and a semicircle, $C$, defined by $z = Re^{j\gamma}$ for $\gamma \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then, if $C$ is followed clockwise,*

$$I_R \triangleq \lim_{R \to \infty} \int_C f(z)dz = \begin{cases} 0 & \text{for } n > 1 \\ -j\pi a & \text{for } n = 1 \end{cases} \tag{C.7.6}$$

*This is proven as follows.*

*On $C$, we have that $z = Re^{j\gamma}$; then*

$$I_R = \lim_{R \to \infty} j \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \ln\left(1 + \frac{a}{R^n}e^{-jn\gamma}\right) Re^{j\gamma} d\gamma \tag{C.7.7}$$

*We also know that*

$$\lim_{|x|\to 0} \ln(1+x) = x \tag{C.7.8}$$

*Then*

$$I_R = \lim_{R\to\infty} \frac{a}{R^{n-1}} j \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} e^{-j(n-1)\gamma} d\gamma \tag{C.7.9}$$

*From this, by evaluation for $n = 1$ and for $n > 1$, the result follows.*

□□□

**Example C.8.** *Consider the function*

$$f(z) = \ln\left(1 + e^{-z\tau}\frac{a}{z^n}\right) \qquad n \geq 1; \quad \tau > 0 \tag{C.7.10}$$

*and a semicircle, $C$, defined by $z = Re^{j\gamma}$ for $\gamma \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then, for clockwise $C$,*

$$I_R \stackrel{\triangle}{=} \lim_{R\to\infty} \int_C f(z)dz = 0 \tag{C.7.11}$$

*This is proven as follows.*
*On $C$, we have that $z = Re^{j\gamma}$; then*

$$I_R = \lim_{R\to\infty} j \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \left[\ln\left(1 + \frac{a}{z^(n+1)}\frac{z}{e^{z\tau}}\right) z\right]_{z=Re^{j\gamma}} d\gamma \tag{C.7.12}$$

*We recall that, if $\tau$ is a positive real number and $\Re\{z\} > 0$, then*

$$\lim_{|z|\to\infty} \frac{z}{e^{z\tau}} = 0 \tag{C.7.13}$$

*Moreover, for very large $R$, we have that*

$$\ln\left(1 + \frac{a}{z^{n+1}}\frac{z}{e^{z\tau}}\right) z\Big|_{z=Re^{j\gamma}} \approx \frac{1}{z^n}\frac{z}{e^{z\tau}}\Big|_{z=Re^{j\gamma}} \tag{C.7.14}$$

*Thus, in the limit, this quantity goes to zero for all positive $n$. The result then follows.*

□□□

**Example C.9.** *Consider the function*

$$f(z) = \ln\left(\frac{z-a}{z+a}\right) \tag{C.7.15}$$

*and a semicircle, $C$, defined by $z = Re^{j\gamma}$ for $\gamma \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then, for clockwise $C$,*

$$I_R \stackrel{\triangle}{=} \lim_{R\to\infty} \int_C f(z)dz = j2\pi a \tag{C.7.16}$$

*This result is obtained by noting that*

$$\ln\left(\frac{z-a}{z+a}\right) = \ln\left(\frac{1-\frac{a}{z}}{1+\frac{a}{z}}\right) = \ln\left(1-\frac{a}{z}\right) - \ln\left(1+\frac{a}{z}\right) \tag{C.7.17}$$

*and then applying the result in example C.7.*

□□□

**Example C.10.** *Consider a function of the form*

$$f(z) = \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \dots \tag{C.7.18}$$

*and $C$, an arc of circle $z = Re^{j\theta}$ for $\theta \in [\theta_1, \theta_2]$. Thus, $dz = jzd\theta$, and*

$$\int_C \frac{dz}{z} = \int_{\theta_1}^{\theta_2} jd\theta = -j(\theta_2 - \theta_1) \tag{C.7.19}$$

*Thus, as $R \to \infty$, we have that*

$$\int_C f(z)dz = -ja_{-1}(\theta_2 - \theta_1) \tag{C.7.20}$$

□□□

**Example C.11.** *Consider, now, $f(z) = z^n$. If the path $C$ is a full circle, centered at the origin and of radius $R$, then*

$$\oint_C z^n dz = \int_{-\pi}^{\pi} \left(R^n e^{jn\theta}\right) jRe^{j\theta} d\theta \tag{C.7.21}$$

$$= \begin{cases} 0 & \text{for } n \neq -1 \\ -2\pi j & \text{for } n = -1 \text{ (integration clockwise)} \end{cases} \tag{C.7.22}$$

□□□

We can now develop Cauchy's Integral Formula.

Say that $f(z)$ can be expanded as

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots \tag{C.7.23}$$

the $a_{-1}$ is called the residue of $f(z)$ at $z_0$.

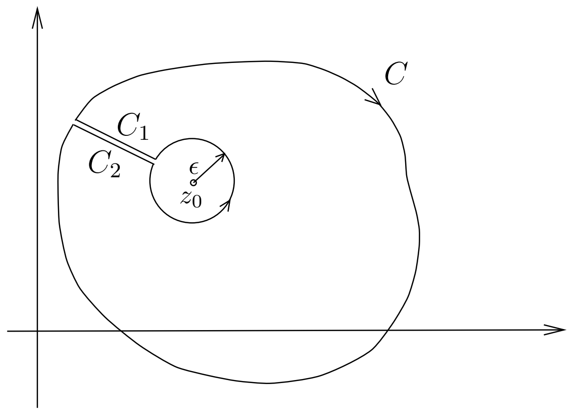


**Figure C.3.** Path for integration of a function having a singularity

Consider the path shown in Figure C.3. Because $f(z)$ is analytic in a region containing $C$, we have that the integral around the complete path shown in Figure C.3 is zero. The integrals along $C_1$ and $C_2$ cancel. The anticlockwise circular integral around $z_0$ can be computed by following example C.11 to yield $2\pi j a_{-1}$. Hence, the integral around the outer curve $C$ is minus the integral around the circle of radius $\epsilon$. Thus,

$$\oint_C f(z)dz = -2\pi j a_{-1} \tag{C.7.24}$$

This leads to the following result.

**Theorem C.8 (Cauchy's Integral Formula).** *Let $g(z)$ be analytic in a region. Let $q$ be a point inside the region. Then $\frac{g(z)}{z-q}$ has residue $g(q)$ at $z = q$, and the integral around any closed contour $C$ enclosing $q$ in a clockwise direction is given by*

$$\oint_C \frac{g(z)}{z - q} dz = -2\pi j g(q) \tag{C.7.25}$$

□□□

We note that the residue of $g(z)$ at an interior point, $z = q$, of a region $D$ can be obtained by integrating $\frac{g(z)}{z-q}$ on the boundary of $D$. Hence, we can determine the value of an analytic function inside a region by its behaviour on the boundary.

## C.8 Poisson and Jensen Integral Formulas

We will next apply the Cauchy Integral formula to develop two related results.

The first result deals with functions that are analytic in the right-half plane (RHP). This is relevant to sensitivity functions in continuous-time systems, where Laplace transforms are used.

The second result deals with functions that are analytic outside the unit disk. This will be a preliminary step to analyzing sensitivity functions in discrete time, on the basis of Z-transforms.

### C.8.1 Poisson's Integral for the Half-Plane

**Theorem C.9.** *Consider a contour $C$ bounding a region $D$. $C$ is a clockwise contour composed by the imaginary axis and a semicircle to the right, centered at the origin and having radius $R \to \infty$. This contour is shown in Figure C.4. Consider some $z_0 = x_0 + jy_0$ with $x_0 > 0$.*

*Let $f(z)$ be a real function of $z$, analytic inside $D$ and of at least the order of $z^{-1}$; $f(z)$ satisfies*

$$\lim_{|z|\to\infty} |z||f(z)| = \beta \qquad 0 \le \beta < \infty \qquad z \in D \tag{C.8.1}$$

*then*

$$f(z_0) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(j\omega)}{j\omega - z_0} d\omega \tag{C.8.2}$$

*Moreover, if (C.8.1) is replaced by the weaker condition*

$$\lim_{|z|\to\infty} \frac{|f(z)|}{|z|} = 0 \qquad z \in D \tag{C.8.3}$$

*then*

$$f(z_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(j\omega)\frac{x_0}{x_0^2 + (y_0 - \omega)^2} d\omega \tag{C.8.4}$$

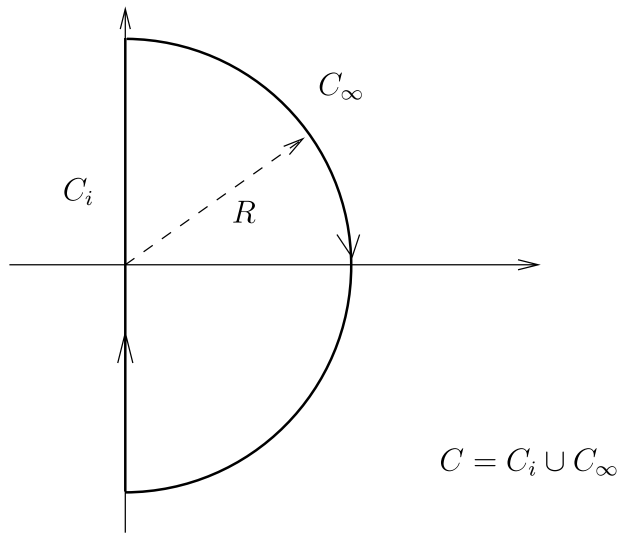


**Figure C.4.** RHP encircling contour

## Proof

Applying Theorem C.8, we have

$$f(z_0) = -\frac{1}{2\pi j} \oint_C \frac{f(z)}{z - z_0} dz = -\frac{1}{2\pi j} \int_{C_i} \frac{f(z)}{z - z_0} dz - \frac{1}{2\pi j} \int_{C_\infty} \frac{f(z)}{z - z_0} dz \quad \text{(C.8.5)}$$

Now, if $f(z)$ satisfies (C.8.1), it behaves like $z^{-1}$ for large $|z|$, i.e., $\frac{f(z)}{z-z_0}$ is like $z^{-2}$. The integral along $C_\infty$ then vanishes and the result (C.8.2) follows.

To prove (C.8.4) when $f(z)$ satisfies (C.8.3), we first consider $z_1$, the image of $z_0$ through the imaginary axis, i.e., $z_1 = -x_0 + jy_0$. Then $\frac{f(z)}{z-z_1}$ is analytic inside $D$, and, on applying Theorem C.7, we have that

$$0 = -\frac{1}{2\pi j} \oint_C \frac{f(z)}{z - z_1} dz \quad \text{(C.8.6)}$$

By combining equations (C.8.5) and (C.8.6), we obtain

$$f(z_0) = -\frac{1}{2j\pi} \oint_C \left( \frac{f(z)}{z - z_0} - \frac{f(z)}{z - z_1} \right) dz = -\frac{1}{2j\pi} \oint_C f(z) \frac{z_0 - z_1}{(z - z_0)(z - z_1)} dz \quad \text{(C.8.7)}$$

Because $C = C_i \cup C_\infty$, the integral over $C$ can be decomposed into the integral along the imaginary axis , $C_i$, and the integral along the semicircle of infinite radius, $C_\infty$. Because $f(z)$ satisfies (C.8.3), this second integral vanishes, because the factor $\frac{z_0 - z_1}{(z-z_0)(z-z_1)}$ is of order $z^{-2}$ at $\infty$.

Then

$$f(z_0) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} f(j\omega)\frac{z_0 - z_1}{(j\omega - z_0)(j\omega - z_1)}d\omega \tag{C.8.8}$$

The result follows upon replacing $z_0$ and $z_1$ by their real; and imaginary-part decompositions.

□□□

**Remark C.1.** *One of the functions that satisfies (C.8.3) but does not satisfy (C.8.1) is $f(z) = \ln g(z)$, where $g(z)$ is a rational function of relative degree $n_r \neq 0$. We notice that, in this case,*

$$\lim_{|z|\to\infty}\left[\frac{|\ln g(z)|}{|z|}\right] = \lim_{R\to\infty}\frac{|K||n_r \ln R + jn_r\theta|}{R} = 0 \tag{C.8.9}$$

*where $K$ is a finite constant and $\theta$ is an angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.*

**Remark C.2.** *Equation (C.8.4) equates two complex quantities. Thus, it also applies independently to their real and imaginary parts. In particular,*

$$\Re\{f(z_0)\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\Re\{f(j\omega)\}\frac{x_0}{x_0^2 + (y_0 - \omega)^2}d\omega \tag{C.8.10}$$

*This observation is relevant to many interesting cases. For instance, when $f(z)$ is as in remark C.1,*

$$\Re\{f(z)\} = \ln|g(z)| \tag{C.8.11}$$

*For this particular case, and assuming that $g(z)$ is a real function of $z$, and that $y_0 = 0$, we have that (C.8.10) becomes*

$$\ln|g(z_0)| = \frac{1}{\pi}\int_{0}^{\infty}\ln|g(j\omega)|\frac{2x_0}{x_0^2 + (y_0 - \omega)^2}d\omega \tag{C.8.12}$$

*where we have used the conjugate symmetry of $g(z)$.*

## C.8.2 Poisson–Jensen Formula for the Half-Plane

**Lemma C.1.** *Consider a function $g(z)$ having the following properties*

*(i) $g(z)$ is analytic on the closed RHP;*

*(ii) $g(z)$ does not vanish on the imaginary axis;*

*(iii) $g(z)$ has zeros in the open RHP, located at $a_1, a_2, \ldots, a_n$;*

*(iv) $g(z)$ satisfies $\lim_{|z|\to\infty} \frac{|\ln g(z)|}{|z|} = 0$.*

*Consider also a point $z_0 = x_0 + jy_0$ such that $x_0 > 0$; then*

$$\ln|g(z_0)| = \sum_{i=1}^{n} \ln\left|\frac{z_0 - a_i}{z_0 + a_i^*}\right| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_0}{x_0^2 + (\omega - y_0)^2} \ln|g(j\omega)|)d\omega \tag{C.8.13}$$

**Proof**

Let

$$\tilde{g}(z) \stackrel{\triangle}{=} g(z)\prod_{i=1}^{n}\frac{z + a_i^*}{z - a_i} \tag{C.8.14}$$

Then, $\ln\tilde{g}(z)$ is analytic within the closed unit disk. If we now apply Theorem C.9 to $\ln\tilde{g}(z)$, we obtain

$$\ln\tilde{g}(z_0) = \ln g(z_0) + \sum_{i=1}^{n}\ln\left(\frac{z_0 + a_i^*}{z_0 - a_i}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x_0}{x_0^2 + (\omega - y_0)^2}\ln\tilde{g}(j\omega)d\omega \tag{C.8.15}$$

We also recall that, if $x$ is any complex number, then $\Re\{\ln x\} = \Re\{\ln|x| + j\angle x\} = \ln|x|$. Thus, the result follows upon equating real parts in the equation above and noting that

$$\ln|\tilde{g}(j\omega)| = \ln|g(j\omega)| \tag{C.8.16}$$

□□□

## C.8.3 Poisson's Integral for the Unit Disk

**Theorem C.10.** *Let $f(z)$ be analytic inside the unit disk. Then, if $z_0 = re^{j\theta}$, with $0 \le r < 1$,*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} P_{1,r}(\theta - \omega) f(e^{j\omega}) d\omega \tag{C.8.17}$$

*where $P_{1,r}(x)$ is the Poisson kernel defined by*

$$P_{\rho,r}(x) \triangleq \frac{\rho^2 - r^2}{\rho^2 - 2r\rho\cos(x) + r^2} \qquad 0 \le r < \rho, \qquad x \in \Re \tag{C.8.18}$$

**Proof**

Consider the unit circle $C$. Then, using Theorem C.8, we have that

$$f(z_0) = \frac{1}{2\pi j}\oint_C \frac{f(z)}{z - z_0} dz \tag{C.8.19}$$

Define

$$z_1 \triangleq \frac{1}{r} e^{j\theta} \tag{C.8.20}$$

Because $z_1$ is outside the region encircled by $C$, the application of Theorem C.8 yields

$$0 = \frac{1}{2\pi j}\oint_C \frac{f(z)}{z - z_1} dz \tag{C.8.21}$$

Subtracting (C.8.21 ) from (C.8.19 ) and changing the variable of integration, we obtain

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{j\omega}) e^{j\omega}\left[\frac{1}{e^{j\omega} - re^{j\theta}} - \frac{r}{re^{j\omega} - e^{j\theta}}\right] d\omega \tag{C.8.22}$$

from which the result follows.

□□□

Consider now a function $g(z)$ which is analytic outside the unit disk. We can then define a function $f(z)$ such that

$$f(z) \triangleq g\left(\frac{1}{z}\right) \tag{C.8.23}$$

Assume that one is interested in obtaining an expression for $g(\zeta_0)$, where $\zeta_0 = re^{j\theta}$, $r > 1$. The problem is then to obtain an expression for $f\left(\frac{1}{\zeta_0}\right)$. Thus, if we define $z_0 \stackrel{\triangle}{=} \frac{1}{\zeta_0} = \frac{1}{r}e^{-j\theta}$, we have, on applying Theorem C.10, that

$$g(\zeta_0) = \frac{1}{2\pi}\int_0^{2\pi} P_{1,\frac{1}{r}}(-\theta - \omega) g(e^{-j\omega}) d\omega \tag{C.8.24}$$

where

$$P_{1,\frac{1}{r}}(-\theta - \omega) = \frac{r^2 - 1}{r^2 - 2rcos(\theta + \omega) + 1} \tag{C.8.25}$$

If, finally, we make the change in the integration variable $\omega = -\nu$, the following result is obtained.

$$g(re^{j\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - 1}{r^2 - 2rcos(\theta - \nu) + 1} g(e^{j\nu}) d\nu \tag{C.8.26}$$

Thus, Poisson's integral for the unit disk can also be applied to functions of a complex variable which are analytic outside the unit circle.

## C.8.4 Poisson–Jensen Formula for the Unit Disk

**Lemma C.2.** *Consider a function $g(z)$ having the following properties:*

*(i)* *$g(z)$ is analytic on the closed unit disk;*

*(ii)* *$g(z)$ does not vanish on the unit circle;*

*(iii)* *$g(z)$ has zeros in the open unit disk, located at $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_{\bar{n}}$.*

*Consider also a point $z_0 = re^{j\theta}$ such that $r < 1$; then*

$$\ln|g(z_0)| = \sum_{i=1}^{\bar{n}} \ln\left|\frac{z_0 - \bar{\alpha}_i}{1 - \bar{\alpha}_i^* z_0}\right| + \frac{1}{2\pi}\int_0^{2\pi} P_{1,r}(\theta - \omega) \ln|g(e^{j\omega})| d\omega \tag{C.8.27}$$

**Proof**

Let

$$\tilde{g}(z) \stackrel{\triangle}{=} g(z) \prod_{i=1}^{n} \frac{1 - \bar{\alpha}_i^* z}{z - \bar{\alpha}_i} \tag{C.8.28}$$

Then $\ln \tilde{g}(z)$ is analytic on the closed unit disk. If we now apply Theorem C.10 to $\ln \tilde{g}(z)$, we obtain

$$\ln \tilde{g}(z_0) = \ln g(z_0) + \sum_{i=1}^{n} \ln \left( \frac{1 - \bar{\alpha}_i^* z_0}{z_0 - \bar{\alpha}_i} \right) = \frac{1}{2\pi} \int_0^{2\pi} P_{1,r}(\theta - \omega) \ln \tilde{g}(e^{j\omega}) d\omega \tag{C.8.29}$$

We also recall that, if $x$ is any complex number, then $\ln x = ln|x| + j\angle x$. Thus the result follows upon equating real parts in the equation above and noting that

$$\ln \left| \tilde{g}(e^{j\omega}) \right| = \ln \left| g(e^{j\omega}) \right| \tag{C.8.30}$$

□□□

**Theorem C.11 (Jensen's formula for the unit disk).** *Let $f(z)$ and $g(z)$ be analytic functions on the unit disk. Assume that the zeros of $f(z)$ and $g(z)$ on the unit disk are $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{\bar{n}}$ and $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_{\bar{m}}$ respectively, where none of these zeros lie on the unit circle.*

*If*

$$h(z) \stackrel{\triangle}{=} z^{\lambda} \frac{f(z)}{g(z)} \qquad \lambda \in \Re \tag{C.8.31}$$

*then*

$$\frac{1}{2\pi} \int_0^{2\pi} \ln |h(e^{j\omega})| d\omega = \ln \left| \frac{f(0)}{g(0)} \right| + \ln \frac{|\bar{\beta}_1 \bar{\beta}_2 \ldots \bar{\beta}_{\bar{m}}|}{|\bar{\alpha}_1 \bar{\alpha}_2 \ldots \bar{\alpha}_{\bar{n}}|} \tag{C.8.32}$$

### Proof

We first note that $\ln |h(z)| = \lambda \ln |z| + \ln |f(z)| - \ln |g(z)|$. We then apply the Poisson–Jensen formula to $f(z)$ and $g(z)$ at $z_0 = 0$ to obtain

$$P_{1,r}(x) = P_{1,0}(x) = 1; \qquad \ln \left| \frac{z_0 - \bar{\alpha}_i}{1 - \bar{\alpha}_i^* z_0} \right| = \ln |\bar{\alpha}_i|; \qquad \ln \left| \frac{z_0 - \bar{\beta}_i}{1 - \bar{\beta}_i^* z_0} \right| = \ln |\bar{\beta}_i| \tag{C.8.33}$$

We thus have that

$$\ln |f(0)| = \sum_{i=1}^{n} \ln |\bar{\alpha}_i| - \frac{1}{2\pi} \int_0^{2\pi} \ln |f(e^{j\omega})| d\omega \tag{C.8.34}$$

$$\ln |g(0)| = \sum_{i=1}^{n} \ln |\bar{\alpha}_i| - \frac{1}{2\pi} \int_0^{2\pi} \ln |g(e^{j\omega})| d\omega \tag{C.8.35}$$

The result follows upon subtracting equation (C.8.35) from (C.8.34), and noting that

$$\frac{\lambda}{2\pi}\int_0^{2\pi} \ln \left|e^{j\omega}\right| d\omega = 0 \tag{C.8.36}$$

□□□

**Remark C.3.** *Further insights can be obtained from equation (C.8.32) if we assume that, in* (C.8.31), $f(z)$ *and* $g(z)$ *are polynomials;*

$$f(z) \;=\; K_f \prod_{i=1}^{n}(z-\alpha_i) \tag{C.8.37}$$

$$g(z) \;=\; \prod_{i=1}^{n}(z-\beta_i) \tag{C.8.38}$$

*then*

$$\left|\frac{f(0)}{g(0)}\right| = |K_f| \left|\frac{\prod_{i=1}^{n}\alpha_i}{\prod_{i=1}^{m}\beta_i}\right| \tag{C.8.39}$$

*Thus,* $\alpha_1, \alpha_2, \ldots \alpha_n$ *and* $\beta_1, \beta_2, \ldots \beta_m$ *are* **all** *the zeros and* **all** *the poles of* $h(z)$, *respectively, that have nonzero magnitude.*

*This allows equation (C.8.32) to be rewritten as*

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|h(e^{j\omega})|d\omega = \ln|K_f| + \ln\frac{|\alpha_1'\alpha_2'\ldots\alpha_{nu}'|}{|\beta_1'\beta_2'\ldots\beta_{mu}'|} \tag{C.8.40}$$

*where* $\alpha_1', \alpha_2', \ldots \alpha_{nu}'$ *and* $\beta_1', \beta_2', \ldots \beta_{mu}'$ *are the zeros and the poles of* $h(z)$, *respectively, that lie outside the unit circle .*

□□□

## C.9 Application of the Poisson–Jensen Formula to Certain Rational Functions

Consider the biproper rational function $\bar{h}(z)$ given by

$$\bar{h}(z) = z^{\bar{\lambda}} \frac{\bar{f}(z)}{\bar{g}(z)} \tag{C.9.1}$$

$\bar{\lambda}$ is a integer number, and $\bar{f}(z)$ and $\bar{g}(z)$ are polynomials of degrees $m_f$ and $m_g$, respectively. Then, due to the biproperness of $\bar{h}(z)$, we have that $\bar{\lambda} + m_f = m_g$.

Further assume that

(i) $\bar{g}(z)$ has no zeros outside the open unit disk,

(ii) $\bar{f}(z)$ does not vanish on the unit circle, and

(iii) $\bar{f}(z)$ vanishes outside the unit disk at $\beta_1, \beta_2, \ldots, \beta_m$.

Define

$$h(z) = \frac{f(z)}{g(z)} \triangleq \bar{h}\left(\frac{1}{z}\right) \tag{C.9.2}$$

where $f(z)$ and $g(z)$ are polynomials.

Then it follows that

(i) $g(z)$ has no zeros in the closed unit disk;

(ii) $f(z)$ does not vanish on the unit circle;

(iii) $f(z)$ vanishes in the open unit disk at $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m$, where $\bar{\beta}_i = \beta_i^{-1}$ for $i = 1, 2, \ldots, \bar{\beta}_m$;

(iv) $h(z)$ is analytic in the closed unit disk;

(v) $h(z)$ does not vanish on the unit circle;

(vi) $h(z)$ has zeros in the open unit disk, located at $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m$.

We then have the following result

**Lemma C.3.** *Consider the function $h(z)$ defined in (C.9.2) and a point $z_0 = re^{j\theta}$ such that $r < 1$; then*

$$\ln|h(z_0)| = \sum_{i=1}^{\bar{m}} \ln\left|\frac{z_0 - \bar{\beta}_i}{1 - \bar{\beta}_i^* z_0}\right| + \frac{1}{2\pi}\int_0^{2\pi} P_{1,r}(\theta - \omega) \ln|h(e^{j\omega})| d\omega \tag{C.9.3}$$

*where $P_{1,r}$ is the Poisson kernel defined in (C.8.18).*

**Proof**

This follows from a straightforward application of Lemma C.2.

□□□

## C.10 Bode's Theorems

We will next review some fundamental results due to Bode.

**Theorem C.12 (Bode integral in the half plane).** *Let $l(z)$ be a proper real, rational function of relative degree $n_r$. Define*

$$g(z) \triangleq (1 + l(z))^{-1} \tag{C.10.1}$$

*and assume that $g(z)$ has neither poles nor zeros in the closed RHP. Then*

$$\int_0^\infty \ln |g(j\omega)| d\omega = \begin{cases} 0 & \text{for } n_r > 1 \\ -\kappa \frac{\pi}{2} & \text{for } n_r = 1 \quad \text{where} \quad \kappa \triangleq \lim_{z\to\infty} z l(z) \end{cases} \tag{C.10.2}$$

**Proof**

Because $\ln g(z)$ is analytic in the closed RHP,

$$\oint_C \ln g(z) dz = 0 \tag{C.10.3}$$

where $C = C_i \cup C_\infty$ is the contour defined in Figure C.4.

Then

$$\oint_C \ln g(z) dz = j \int_{-\infty}^{\infty} \ln g(j\omega) d\omega - \int_{C_\infty} \ln(1 + l(z)) dz \tag{C.10.4}$$

For the first integral on the right-hand side of equation (C.10.4), we use the conjugate symmetry of $g(z)$ to obtain

$$\int_{-\infty}^{\infty} \ln g(j\omega) d\omega = 2 \int_0^\infty \ln |g(j\omega)| d\omega \tag{C.10.5}$$

For the second integral, we notice that, on $C_\infty$, $l(z)$ can be approximated by

$$\frac{a}{z^{n_r}} \tag{C.10.6}$$

The result follows upon using example C.7 and noticing that $a = \kappa$ for $n_r = 1$.

□□□

**Remark C.4.** *If $g(z) = (1 + e^{-z\tau} l(z))^{-1}$ for $\tau > 0$, then result (C.10.9) becomes*

$$\int_0^\infty \ln|g(j\omega)|d\omega = 0 \qquad \forall n_r > 0 \tag{C.10.7}$$

*The proof of (C.10.7) follows along the same lines as those of Theorem C.12 and by using the result in example C.8.*

**Theorem C.13 (Modified Bode integral).** *Let $l(z)$ be a proper real, rational function of relative degree $n_r$. Define*

$$g(z) \stackrel{\triangle}{=} (1 + l(z))^{-1} \tag{C.10.8}$$

*Assume that $g(z)$ is analytic in the closed RHP and that it has q zeros in the open RHP, located at $\zeta_1, \zeta_2, \dots, \zeta_q$ with $\Re(\zeta_i) > 0$. Then*

$$\int_0^\infty \ln|g(j\omega)|d\omega = \begin{cases} \pi \sum_{i=1}^{q} \zeta_i & \textit{for } n_r > 1 \\ -\kappa\frac{\pi}{2} + \pi \sum_{i=1}^{q} \zeta_i & \textit{for } n_r = 1 \quad \textit{where} \quad \kappa \stackrel{\triangle}{=} \lim_{z\to\infty} zl(z) \end{cases} \tag{C.10.9}$$

**Proof**

We first notice that $\ln g(z)$ is no longer analytic on the RHP. We then define

$$\tilde{g}(z) \stackrel{\triangle}{=} g(z) \prod_{i=1}^{q} \frac{z + \zeta_i}{z - \zeta_i} \tag{C.10.10}$$

Thus, $\ln \tilde{g}(z)$ is analytic in the closed RHP. We can then apply Cauchy's integral in the contour $C$ described in Figure C.4 to obtain

$$\oint_C \ln \tilde{g}(z)dz = 0 = \oint_C \ln g(z)dz + \sum_{i=1}^{q} \oint_C \ln \frac{z + \zeta_i}{z - \zeta_i} dz \tag{C.10.11}$$

The first integral on the right-hand side can be expressed as

$$\oint_C \ln g(z)dz = 2j \int_0^\infty \ln|g(j\omega)|d\omega + \int_{C_\infty} \ln g(z)dz \tag{C.10.12}$$

where, by using example C.7.

$$\int_{C_\infty} \ln g(z)dz = \begin{cases} 0 & \text{for } n_r > 1 \\ j\kappa\pi & \text{for } n_r = 1 \quad \text{where} \quad \kappa \stackrel{\triangle}{=} \lim_{z\to\infty} zl(z) \end{cases} \tag{C.10.13}$$

The second integral on the right-hand side of equation (C.10.11) can be computed as follows:

$$\oint_C \ln \frac{z+\zeta_i}{z-\zeta_i} dz = j\int_{-\infty}^{\infty} \ln \frac{j\omega+\zeta_i}{j\omega-\zeta_i} d\omega + \int_{C_\infty} \ln \frac{z+\zeta_i}{z-\zeta_i} dz \qquad \text{(C.10.14)}$$

We note that the first integral on the right-hand side is zero, and by using example C.9, the second integral is equal to $-2j\pi\zeta_i$. Thus, the result follows.

□□□

**Remark C.5.** *Note that $g(z)$ is a real function of $z$, so*

$$\sum_{i=1}^{q} \zeta_i = \sum_{i=1}^{q} \Re\{\zeta_i\} \qquad \text{(C.10.15)}$$

□□□

**Remark C.6.** *If $g(z) = \left(1 + e^{-z\tau} l(z)\right)^{-1}$ for $\tau > 0$, then the result (C.10.9) becomes*

$$\int_0^{\infty} \ln |g(j\omega)| d\omega = \pi \sum_{i=1}^{q} \Re\{\zeta_i\} \qquad \forall n_r > 0 \qquad \text{(C.10.16)}$$

*The proof of (C.10.16) follows along the same lines as those of Theorem C.13 and by using the result in example C.8.*

**Remark C.7.** *The Poisson, Jensen, and Bode formulae assume that a key function is analytic, not only inside a domain $D$, but also on its border $C$. Sometimes, there may exist singularities on $C$. These can be dealt with by using an infinitesimal circular indentation in $C$, constructed so as to leave the singularity outside $D$. For the functions of interest to us, the integral along the indentation vanishes. This is illustrated in example C.6 for a logarithmic function, when $D$ is the right-half plane and there is a singularity at the origin.*

□□□

Appendix D

# PROPERTIES OF CONTINUOUS-TIME RICCATI EQUATIONS

This appendix summarizes key properties of the Continuous-Time Differential Riccati Equation (CTDRE);

$$\frac{d\mathbf{P}}{dt} = -\mathbf{A}^T\mathbf{P}(t) - \mathbf{P}(t)\mathbf{A} + \mathbf{P}(t)\mathbf{B}\boldsymbol{\Phi}^{-1}\mathbf{B}^T\mathbf{P}(t) - \boldsymbol{\Psi} \tag{D.0.1}$$

$$\mathbf{P}(t_f) = \boldsymbol{\Psi}_f \tag{D.0.2}$$

and the Continuous-Time Algebraic Riccati Equation (CTARE)

$$0 = -\mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{A} + \mathbf{P}\mathbf{B}\boldsymbol{\Phi}^{-1}\mathbf{B}^T\mathbf{P} - \boldsymbol{\Psi} \tag{D.0.3}$$

## D.1 Solutions of the CTDRE

The following lemma gives a useful alternative expression for $\mathbf{P}(t)$.

**Lemma D.1.** *The solution, $\mathbf{P}(t)$, to the CTDRE* (D.0.1)*, can be expressed as*

$$\mathbf{P}(t) = \mathbf{N}(t)[\mathbf{M}(t)]^{-1} \tag{D.1.1}$$

*where $\mathbf{M}(t) \in \mathbb{R}^{n\times n}$ and $\mathbf{N}(t) \in \mathbb{R}^{n\times n}$ satisfy the following equation:*

$$\frac{d}{dt}\begin{bmatrix}\mathbf{M}(t)\\ \mathbf{N}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A} & -\mathbf{B}\boldsymbol{\Phi}^{-1}\mathbf{B}^T\\ -\boldsymbol{\Psi} & -\mathbf{A}^T\end{bmatrix}\begin{bmatrix}\mathbf{M}(t)\\ \mathbf{N}(t)\end{bmatrix} \tag{D.1.2}$$

*subject to*

$$\mathbf{N}(t_f)[\mathbf{M}(t_f)]^{-1} = \boldsymbol{\Psi}_f \tag{D.1.3}$$

**Proof**

*We show that* $\mathbf{P}(t)$*, as defined above, satisfies the CTDRE. We first have that*

$$\frac{d\mathbf{P}(t)}{dt} = \frac{d\mathbf{N}(t)}{dt}[\mathbf{M}(t)]^{-1} + \mathbf{N}(t)\frac{d[\mathbf{M}(t)]^{-1}}{dt} \tag{D.1.4}$$

*The derivative of* $[\mathbf{M}(t)]^{-1}$ *can be computed by noting that* $\mathbf{M}(t)[\mathbf{M}(t)]^{-1} = \mathbf{I}$*; then*

$$\frac{d\mathbf{I}}{dt} = \mathbf{0} = \frac{d\mathbf{M}(t)}{dt}[\mathbf{M}(t)]^{-1} + \mathbf{M}(t)\frac{d[\mathbf{M}(t)]^{-1}}{dt} \tag{D.1.5}$$

*from which we obtain*

$$\frac{d[\mathbf{M}(t)]^{-1}}{dt} = -[\mathbf{M}(t)]^{-1}\frac{d\mathbf{M}(t)}{dt}[\mathbf{M}(t)]^{-1} \tag{D.1.6}$$

*Thus, equation* (D.1.4) *can be used with* (D.1.2) *to yield*

$$\begin{aligned} -\frac{d\mathbf{P}(t)}{dt} &= \mathbf{A}^T\mathbf{N}(t)[\mathbf{M}(t)]^{-1} + \mathbf{N}(t)[\mathbf{M}(t)]^{-1}\mathbf{A} + \mathbf{\Psi} \\ &\quad - \mathbf{N}(t)[\mathbf{M}(t)]^{-1}\mathbf{B}[\Phi]^{-1}\mathbf{B}^T\mathbf{N}(t)[\mathbf{M}(t)]^{-1} \end{aligned} \tag{D.1.7}$$

*which shows that* $\mathbf{P}(t)$ *also satisfies* (D.0.1)*, upon using* (D.1.1).

The matrix on the right-hand side of (D.1.2), namely,

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{\Phi}^{-1}\mathbf{B}^T \\ -\mathbf{\Psi} & -\mathbf{A}^T \end{bmatrix} \qquad \mathbf{H} \in \mathbb{R}^{2n\times 2n} \tag{D.1.8}$$

is called the *Hamiltonian* matrix associated with this problem.

Next, note that (D.0.1) can be expressed in compact form as

$$\begin{bmatrix} -\mathbf{P}(t) & \mathbf{I} \end{bmatrix} \mathbf{H} \begin{bmatrix} \mathbf{I} \\ \mathbf{P}(t) \end{bmatrix} = \frac{d\mathbf{P}(t)}{dt} \tag{D.1.9}$$

Then, not surprisingly, solutions to the CTDRE, (D.0.1), are intimately connected to the properties of the Hamiltonian matrix.

We first note that $\mathbf{H}$ has the following reflexive property:

$$\mathbf{H} = -\mathbf{T}\mathbf{H}^T\mathbf{T}^{-1} \qquad \text{with} \qquad \mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{I_n} \\ -\mathbf{I_n} & \mathbf{0} \end{bmatrix} \tag{D.1.10}$$

where $\mathbf{I_n}$ is the identity matrix in $\mathbb{R}^{n\times n}$.

Recall that a similarity transformation preserves the eigenvalues; thus, the eigenvalues of $\mathbf{H}$ are the same as those of $-\mathbf{H}^T$. On the other hand, the eigenvalues of $\mathbf{H}$ and $\mathbf{H}^T$ must be the same. Hence, the spectral set of $\mathbf{H}$ is the union of two sets, $\Lambda_s$ and $\Lambda_u$, such that, if $\lambda \in \Lambda_s$, then $-\lambda \in \Lambda_u$. We assume that $\mathbf{H}$ does not contain any eigenvalue on the imaginary axis (note that it suffices, for this to occur, that $(\mathbf{A}, \mathbf{B})$ be stabilizable and that the pair $(\mathbf{A}, \mathbf{\Psi}^{\frac{1}{2}})$ have no undetectable poles on the stability boundary). In this case, $\Lambda_s$ can be so formed that it contains only the eigenvalues of $\mathbf{H}$ that lie in the open LHP. Then, there always exists a nonsingular transformation $\mathbf{V} \in \mathbb{R}^{2n\times 2n}$ such that

$$[\mathbf{V}]^{-1}\mathbf{H}\mathbf{V} = \begin{bmatrix} \mathbf{H_s} & \mathbf{0} \\ \mathbf{0} & \mathbf{H_u} \end{bmatrix} \tag{D.1.11}$$

where $\mathbf{H_s}$ and $\mathbf{H_u}$ are diagonal matrices with eigenvalue sets $\Lambda_s$ and $\Lambda_u$, respectively.

We can use $\mathbf{V}$ to transform the matrices $\mathbf{M}(t)$ and $\mathbf{N}(t)$, to obtain

$$\begin{bmatrix} \tilde{\mathbf{M}}(t) \\ \tilde{\mathbf{N}}(t) \end{bmatrix} = [\mathbf{V}]^{-1} \begin{bmatrix} \mathbf{M}(t) \\ \mathbf{N}(t) \end{bmatrix} \tag{D.1.12}$$

Thus, (D.1.2) can be expressed in the equivalent form:

$$\frac{d}{dt}\begin{bmatrix} \tilde{\mathbf{M}}(t) \\ \tilde{\mathbf{N}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{H_s} & \mathbf{0} \\ \mathbf{0} & \mathbf{H_u} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{M}}(t) \\ \tilde{\mathbf{N}}(t) \end{bmatrix} \tag{D.1.13}$$

If we partition $\mathbf{V}$ in a form consistent with the matrix equation (D.1.13), we have that

$$\mathbf{V} = \begin{bmatrix} \mathbf{V_{11}} & \mathbf{V_{12}} \\ \mathbf{V_{21}} & \mathbf{V_{22}} \end{bmatrix} \tag{D.1.14}$$

The solution to the CTDRE is then given by the following lemma.

**Lemma D.2.** *A solution for equation* (D.0.1) *is given by*

$$\mathbf{P}(t) = \mathbf{P_1}(t)[\mathbf{P_2}(t)]^{-1} \tag{D.1.15}$$

*where*

$$\mathbf{P_1}(t) = \mathbf{V_{21}} + \mathbf{V_{22}}e^{-\mathbf{H_u}(t_f-t)}\mathbf{V_a}e^{\mathbf{H_s}(t_f-t)} \tag{D.1.16}$$

$$\mathbf{P_2}(t) = \left[\mathbf{V_{11}} + \mathbf{V_{12}}e^{-\mathbf{H_u}(t_f-t)}\mathbf{V_a}e^{\mathbf{H_s}(t_f-t)}\right]^{-1} \tag{D.1.17}$$

$$\mathbf{V_a} \stackrel{\triangle}{=} -[\mathbf{V_{22}} - \mathbf{\Psi}_f\mathbf{V_{12}}]^{-1}[\mathbf{V_{21}} - \mathbf{\Psi}_f\mathbf{V_{11}}] = \tilde{\mathbf{N}}(t_f)\left[\tilde{\mathbf{M}}(t_f)\right]^{-1} \tag{D.1.18}$$

**Proof**

*From (D.1.12), we have*

$$\begin{aligned}\mathbf{M}(t_f) &= \mathbf{V}_{11}\tilde{\mathbf{M}}(t_f) + \mathbf{V}_{12}\tilde{\mathbf{N}}(t_f) \\ \mathbf{N}(t_f) &= \mathbf{V}_{21}\tilde{\mathbf{M}}(t_f) + \mathbf{V}_{22}\tilde{\mathbf{N}}(t_f)\end{aligned} \tag{D.1.19}$$

*Hence, from (D.1.3),*

$$\left[\mathbf{V}_{21}\tilde{\mathbf{M}}(t_f) + \mathbf{V}_{22}\tilde{\mathbf{N}}(t_f)\right]\left[\mathbf{V}_{11}\tilde{\mathbf{M}}(t_f) + \mathbf{V}_{12}\tilde{\mathbf{N}}(t_f)\right]^{-1} = \mathbf{\Psi}_f \tag{D.1.20}$$

*or*

$$\left[\mathbf{V}_{21} + \mathbf{V}_{22}\tilde{\mathbf{N}}(t_f)[\tilde{\mathbf{M}}(t_f)]^{-1}\right]\left[\mathbf{V}_{11} + \mathbf{V}_{12}\tilde{\mathbf{N}}(t_f)[\tilde{\mathbf{M}}(t_f)]^{-1}\right]^{-1} = \mathbf{\Psi}_f \tag{D.1.21}$$

*or*

$$\tilde{\mathbf{N}}(t_f)[\tilde{\mathbf{M}}(t_f)]^{-1} = -\left[\mathbf{V}_{22}\Psi_f\mathbf{V}_{12}\right]^{-1}\left[\mathbf{V}_{21} - \Psi_f\mathbf{V}_{11}\right] = \mathbf{V_a} \tag{D.1.22}$$

*Now, from (D.1.10),*

$$\begin{aligned}\mathbf{P}(t) &= \mathbf{N}(t)[\mathbf{M}(t)]^{-1} \\ &= \left[\mathbf{V}_{21}\tilde{\mathbf{M}}(t) + \mathbf{V}_{22}\tilde{\mathbf{N}}(t)\right]\left[\mathbf{V}_{11}\tilde{\mathbf{M}}(t) + \mathbf{V}_{12}\tilde{\mathbf{N}}(t)\right]^{-1} \\ &= \left[\mathbf{V}_{21} + \mathbf{V}_{22}\tilde{\mathbf{N}}(t)[\tilde{\mathbf{M}}(t)]^{-1}\right]\left[\mathbf{V}_{11} + \mathbf{V}_{12}\tilde{\mathbf{N}}(t)[\tilde{\mathbf{M}}(t)]^{-1}\right]^{-1}\end{aligned} \tag{D.1.23}$$

*and the solution to (D.1.13) is*

$$\begin{aligned}\tilde{\mathbf{M}}(t_f) &= e^{\mathbf{H}_s(t_f - t)}\tilde{\mathbf{M}}(t) \\ \tilde{\mathbf{N}}(t_f) &= e^{\mathbf{H}_u(t_f - t)}\tilde{\mathbf{N}}(t)\end{aligned} \tag{D.1.24}$$

*Hence,*

$$\tilde{\mathbf{N}}(t)[\tilde{\mathbf{M}}(t)]^{-1} = e^{-\mathbf{H}_u(t_f - t)}\tilde{\mathbf{N}}(t_f)[\tilde{\mathbf{M}}(t_f)]^{-1}e^{\mathbf{H}_s(t_f - t)} \tag{D.1.25}$$

*Substituting (D.1.25) into (D.1.23) gives the result.*

□□□

## D.2 Solutions of the CTARE

The **C**ontinuous **T**ime **A**lgebraic **R**iccati **E**quation (CTARE) has many solutions, because it is a matrix quadratic equation. The solutions can be characterized as follows.

**Lemma D.3.** *Consider the following CTARE:*

$$0 = \mathbf{\Psi} - \mathbf{PB}\mathbf{\Phi}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{PA} + \mathbf{A}^T\mathbf{P} \tag{D.2.1}$$

*(i) The CTARE can be expressed as*

$$\begin{bmatrix} -\mathbf{P} & \mathbf{I} \end{bmatrix} \mathbf{H} \begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix} = \mathbf{0} \tag{D.2.2}$$

*where* $\mathbf{H}$ *is defined in (D.1.8).*

*(ii) Let* $\overline{\mathbf{V}}$ *be defined so that*

$$\overline{\mathbf{V}}^{\,-1}\mathbf{H}\overline{\mathbf{V}} = \begin{bmatrix} \mathbf{\Lambda}_a & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_b \end{bmatrix} \tag{D.2.3}$$

*where* $\mathbf{\Lambda}_a, \mathbf{\Lambda}_b$ *are any partitioning of the (generalized) eigenvalues of* $\mathbf{H}$ *such that, if* $\lambda$ *is equal to* $(\Lambda_a)_i$ *for same* $i$*, then* $-\lambda^* = (\Lambda_b)_j$ *for some* $j$*.*

*Let*

$$\overline{\mathbf{V}} = \begin{bmatrix} \overline{\mathbf{V}}_{11} & \overline{\mathbf{V}}_{12} \\ \overline{\mathbf{V}}_{21} & \overline{\mathbf{V}}_{22} \end{bmatrix} \tag{D.2.4}$$

*Then* $\overline{\mathbf{P}} = \overline{\mathbf{V}}_{21}\overline{\mathbf{V}}_{11}^{\,-1}$ *is a solution of the CTARE.*

**Proof**

*(i) This follows direct substitution.*

*(ii) The form of* $\overline{\mathbf{P}}$ *ensures that*

$$\begin{bmatrix} -\overline{\mathbf{P}} & \mathbf{I} \end{bmatrix} \overline{\mathbf{V}} = \begin{bmatrix} \mathbf{0} & * \end{bmatrix} \tag{D.2.5}$$

$$\overline{\mathbf{V}}^{-1} \begin{bmatrix} \overline{\mathbf{P}} \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} * \\ \mathbf{0} \end{bmatrix} \tag{D.2.6}$$

*where * denotes a possible nonzero component.*
*Hence,*

$$\begin{bmatrix}-\overline{\mathbf{P}} & \mathbf{I}\end{bmatrix} \overline{\mathbf{V}}\mathbf{\Lambda}\mathbf{V}^{-1} \begin{bmatrix}\mathbf{P} \\ \mathbf{I}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & *\end{bmatrix} \mathbf{\Lambda} \begin{bmatrix} * \\ \mathbf{0}\end{bmatrix} \tag{D.2.7}$$

$$= 0 \tag{D.2.8}$$

□□□

## D.3 The stabilizing solution of the CTARE

We see from Section D.2 that we have as many solutions to the CTARE as there are ways of partitioning the eigevalues of H into the groups $\mathbf{\Lambda}_a$ and $\mathbf{\Lambda}_b$. Provided that $(A, B)$ is stabilizable and that $(\mathbf{\Psi}^{\frac{1}{2}}, \mathbf{A})$ has no unobservable modes in the imaginary axis, then $\mathbf{H}$ has no eigenvalues in the imaginary axis. In this case, there exists a unique way of partitioning the eigenvalues so that $\mathbf{\Lambda}_a$ contains only the stable eigenvalues of $\mathbf{H}$. We call the corresponding (unique) solution of the CTARE *the stabilizing solution* and denote it by $\mathbf{P}^s_\infty$.
Properties of the stabilizing solution are given in the following.

**Lemma D.4.** *(a) The stabilizing solution has the property that the closed loop* $\mathbf{A}$ *matrix,*

$$\mathbf{A_{cl}} = \mathbf{A} - \mathbf{B}\mathbf{K}^s_\infty \tag{D.3.1}$$

*where*

$$K^s_\infty = \Phi^{-1} B^T \mathbf{P}^s_\infty \tag{D.3.2}$$

*has eigenvalues in the open left-half plane.*

*(b) If* $(\mathbf{\Psi}^{\frac{1}{2}}, \mathbf{A})$ *is detectable, then the stabilizing solution is the only nonnegative solution of the CTARE.*

*(c) If* $(\mathbf{\Psi}^{\frac{1}{2}}, \mathbf{A})$ *has no unobservable modes inside the stability boundary, then the stabilizing solution is positive definite, and conversely.*

*(d) If* $(\Psi^{\frac{1}{2}}, A)$ *has an unobservable mode outside the stabilizing region, then in addition to the stabilizing solution, there exists at least one other nonnegative solution of the CTARE. However, the stabilizing solution,* $\mathbf{P}^s_\infty$ *has the property that*

$$\mathbf{P}^s_\infty - \mathbf{P}'_\infty \geq \mathbf{0} \tag{D.3.3}$$

*where* $\mathbf{P}'_\infty$ *is any other solution of the CTARE.*

**Proof**

*For part (a), we argue as follows:*

*Consider ( D.1.11) and (D.1.14). Then*

$$\mathbf{H}\begin{bmatrix}\mathbf{V_{11}}\\\mathbf{V_{21}}\end{bmatrix} = \begin{bmatrix}\mathbf{V_{11}}\\\mathbf{V_{21}}\end{bmatrix}\mathbf{H_s} \tag{D.3.4}$$

*which implies that*

$$\mathbf{H}\begin{bmatrix}\mathbf{I}\\\mathbf{V_{21}}\mathbf{V_{11}}^{-1}\end{bmatrix} = \mathbf{H}\begin{bmatrix}\mathbf{I}\\\mathbf{P}_\infty\end{bmatrix} = \begin{bmatrix}\mathbf{V_{11}}\mathbf{H_s}\mathbf{V_{11}}^{-1}\\\mathbf{V_{21}}\mathbf{H_s}\mathbf{V_{11}}^{-1}\end{bmatrix} \tag{D.3.5}$$

*If we consider only the first row in (D.3.5), then, using (D.1.8), we have*

$$\mathbf{V_{11}}\mathbf{H_s}\mathbf{V_{11}}^{-1} = \mathbf{A} - \mathbf{B}\mathbf{\Phi}^{-1}\mathbf{B}^T\mathbf{P}_\infty = \mathbf{A} - \mathbf{BK} \tag{D.3.6}$$

*Hence, the closed-loop poles are the eigenvalues of* $\mathbf{H_s}$ *and, by construction, these are stable.*

*We leave the reader to pursue parts (b), (c), and (d) by studying the references given at the end of Chapter 24.*

□□□

## D.4 Convergence of Solutions of the CTARE to the Stabilizing Solution of the CTARE

Finally, we show that, under reasonable conditions, the solution of the CTDRE will converge to the unique stabilizing solution of the CTARE. In the sequel, we will be particularly interested in the stabilizing solution to the CTARE.

**Lemma D.5.** *Provided that* $(A, B)$ *is stabilizable and that* $(\mathbf{\Psi}^{\frac{1}{2}}, \mathbf{A})$ *has no unobservable poles on the imaginary axis and that* $\mathbf{\Psi}_f > \mathbf{P}_\infty^s$*, then*

$$\lim_{t_f\to\infty}\mathbf{P}(t) = \mathbf{P}_\infty^s \tag{D.4.1}$$

**Proof**

*We observe that the eigenvalues of* $\mathbf{H}$ *can be grouped so that* $\mathbf{\Lambda}_s$ *contains only eigenvalues that lie in the left-half plane. We then have that*

$$\lim_{t_f\to\infty} e^{\mathbf{H_s}(t_f-t)} = \mathbf{0} \qquad \text{and} \qquad \lim_{t_f\to\infty} e^{-\mathbf{H_u}(t_f-t)} = \mathbf{0} \tag{D.4.2}$$

*given that* $\mathbf{H}_s$ *and* $-\mathbf{H}_u$ *are matrices with eigenvalues strictly inside the LHP.*

*The result then follows from (D.1.16) to (D.1.17).*

**Remark D.1.** *Actually, provided that* $(\Psi^{\frac{1}{2}}, A)$ *is detectable, then it suffices to have* $\mathbf{\Psi}_f \geq \mathbf{0}$ *in Lemma D.5*

□□□

## D.5 Duality between Linear Quadratic Regulator and Optimal Linear Filter

The close connections between the optimal filter and the LQR problem can be expressed directly as follows: We consider the problem of estimating a particular linear combination of the states, namely,

$$z(t) = f^T x(t) \tag{D.5.1}$$

(The final solution will turn out to be independent of $f$, and thus will hold for the complete state vector.)

Now we will estimate $z(t)$ by using a linear filter of the following form:

$$\hat{z}(t) = \int_0^t h(t-\tau)^T y'(\tau) d\tau + g^T \hat{x}_o \tag{D.5.2}$$

where $h(t)$ is the impulse response of the filter and where $\hat{x}_o$ is a given estimate of the initial state. Indeed, we will assume that (22.10.17) holds, that is, that the initial state $x(0)$ satisfies

$$\mathcal{E}(x(0) - \hat{x}_o)(x(0) - \hat{x}_o)^T = \mathbf{P_o} \tag{D.5.3}$$

We will be interested in designing the filter impulse response, $h(\tau)$, so that $\hat{z}(t)$ is *close* to $z(t)$ in some sense. (Indeed, the precise sense we will use is a quadratic form.) From (D.5.1) and (D.5.2), we see that

$$\begin{aligned} \tilde{z}(t) &= z(t) - \hat{z}(t) \\ &= f^T x(t) - \int_0^t h(t-\tau)^T y'(\tau) d\tau - g^T \hat{x}_o \\ &= f^T x(t) - \int_0^t h(t-\tau)^T \big(\mathbf{C} x(\tau) + \dot{v}(t)\big) d\tau - g^T \hat{x}_o \end{aligned} \tag{D.5.4}$$

Equation (D.5.4) is somewhat difficult to deal with, because of the cross-product between $h(t-\tau)$ and $x(t)$ in the integral. Hence, we introduce another variable, $\lambda$, by using the following equation

$$\frac{d\lambda(\tau)}{d\tau} = -\mathbf{A}^T \lambda(\tau) - \mathbf{C}^T u(\tau) \tag{D.5.5}$$

$$\lambda(t) = -f \tag{D.5.6}$$

where $u(\tau)$ is the reverse time form of $h$:

$$u(\tau) = h(t - \tau) \tag{D.5.7}$$

Substituting (D.5.5) into (D.5.4) gives

$$\begin{aligned}\tilde{z}(t) =& f^T x(t) + \int_0^t \left[\frac{d\lambda(\tau)}{d\tau} + \mathbf{A}^T \lambda(\tau)\right]^T x(\tau) d\tau \\ &- \int_0^t u(\tau)\dot{v}(\tau) d\tau - g^T \hat{x}_o\end{aligned} \tag{D.5.8}$$

Using integration by parts, we then obtain

$$\begin{aligned}\tilde{z}(t) =& f^T x(t) + \left[\lambda^T x(\tau)\right]_0^t - g^T \hat{x}_o \\ &+ \int_0^t \left(-\lambda(\tau)^T \frac{dx(\tau)}{d\tau} + \lambda(\tau)^T \mathbf{A} x(\tau) - u(\tau)^T \frac{dv(\tau)}{d\tau}\right) d\tau\end{aligned} \tag{D.5.9}$$

Finally, using (22.10.5) and (D.5.6), we obtain

$$\begin{aligned}\tilde{z}(t) =& \lambda(0)^T (x(0) - \hat{x}_o) + \int_0^t \left(-\lambda(\tau)^T \frac{dw(\tau)}{d\tau} - u(\tau)^T \frac{dv(\tau)}{d\tau}\right) d\tau \\ &- (\lambda(0) + g)^T \hat{x}_o\end{aligned} \tag{D.5.10}$$

Hence, squaring and taking mathematical expectations, we obtain (upon using (D.5.3), (22.10.3), and (22.10.4) ) the following:

$$\begin{aligned}\mathcal{E}\{\tilde{z}(t)^2\} =& \lambda(0)^T \mathbf{P_o} \lambda(0) + \int_0^t \big(\lambda(\tau)^T \mathbf{Q} \lambda(\tau) + u(\tau)^T \mathbf{R} u(\tau)\big) d\tau \\ &+ \parallel (\lambda(0) + g)^T \hat{x}_o \parallel^2\end{aligned} \tag{D.5.11}$$

The last term in (D.5.11) is zero if $g = -\lambda(0)$. Thus, we see that the design of the optimal linear filter can be achieved by minimizing

$$J = \lambda(0)^T \mathbf{P_o} \lambda(0) + \int_0^t \big(\lambda(\tau)^T \mathbf{Q} \lambda(\tau) + u(\tau)^T \mathbf{R} u(\tau)\big) d\tau \tag{D.5.12}$$

where $\lambda(\tau)$ satisfies the reverse-time equations (D.5.5) and (D.5.6).

We recognize the set of equations formed by (D.5.5), (D.5.6), and (D.5.12) as a **standard linear regulator problem**, provided that the connections shown in Table D.1 are made.

Finally, by using the (dual) optimal control results presented earlier, we see that the optimal filter is given by

| Regulator | Filter | Regulator | Filter |
|---|---|---|---|
| $\tau$ | $t-\tau$ | $t_f$ | $0$ |
| $\mathbf{A}$ | $-\mathbf{A}^T$ | $\mathbf{\Psi}$ | $\mathbf{Q}$ |
| $\mathbf{B}$ | $-\mathbf{C}^T$ | $\mathbf{\Phi}$ | $\mathbf{R}$ |
| $x$ | $\lambda$ | $\mathbf{\Psi}_f$ | $\mathbf{P_o}$ |

**Table D.1.** Duality in quadratic regulators and filters

$$\hat{z}^o(\tau) = \int_o^t u^o(\tau)^T y'(\tau) d\tau + g^T \hat{x}_o \tag{D.5.13}$$

where

$$u^o(\tau) = -\mathbf{K_f}(\tau)\lambda(\tau) \tag{D.5.14}$$
$$\mathbf{K_f}(\tau) = \mathbf{R}^{-1}\mathbf{C}\mathbf{\Sigma}(\tau) \tag{D.5.15}$$

and $\mathbf{\Sigma}(\tau)$ satisfies the dual form of (D.0.1), (22.4.18):

$$-\frac{d\mathbf{\Sigma}(t)}{dt} = \mathbf{Q} - \mathbf{\Sigma}(t)\mathbf{C}^T\mathbf{R}^{-1}\mathbf{C}\mathbf{\Sigma}(t) + \mathbf{\Sigma}(t)\mathbf{A}^T + \mathbf{A}\mathbf{\Sigma}(t) \tag{D.5.16}$$

$$\mathbf{\Sigma}(0) = \mathbf{P_o} \tag{D.5.17}$$

Substituting (D.5.14) into (D.5.5), (D.5.6) we see that

$$\frac{d\lambda(\tau)}{d\tau} = -\mathbf{A}^T\lambda(\tau) + \mathbf{C}^T\mathbf{K_f}(\tau)\lambda(\tau) \tag{D.5.18}$$
$$\lambda(t) = -f \tag{D.5.19}$$
$$u^o(\tau) = -\mathbf{K_f}(\tau)\lambda(\tau) \tag{D.5.20}$$
$$g = -\lambda(0) \tag{D.5.21}$$

We see that $u^o(\tau)$ is the output of a linear homogeneous equation. Let $\nu = (t-\tau)$, and define $\mathbf{\Phi}(\nu)$ as the state transition matrix from $\tau = 0$ for the time-varying system having $A - matrix$ equal to $\left[\mathbf{A} - \mathbf{K}_f(t-\nu)^T\mathbf{C}\right]$. Then

$$
\begin{aligned}
\lambda(\tau) &= -\mathbf{\Phi}(t-\tau)^T f \\
\lambda(0) &= -\mathbf{\Phi}(t)^T f \\
u^0(\tau) &= \mathbf{K_f}(\tau)\mathbf{\Phi}(t-\tau)^T f
\end{aligned}
\tag{D.5.22}
$$

Hence, the optimal filter satisfies

$$
\begin{aligned}
\hat{z}(t) &= g^T \hat{x}_o + \int_0^t u^o y'(\tau) d\tau \\
&= -\lambda(0)^T \hat{x}_o + \int_0^t f^T \mathbf{\Phi}(t-\tau) \mathbf{K_f}^T(\tau) y'(\tau) d\tau \\
&= f^T \left( \mathbf{\Phi}(t)\hat{x}_o + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{K_f}^T(\tau) y'(\tau) d\tau \right) \\
&= f^T \hat{x}(t)
\end{aligned}
\tag{D.5.23}
$$

where

$$
\hat{x}(t) = \mathbf{\Phi}(t)\hat{x}_o + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{K_f}^T(\tau) y'(\tau) d\tau \tag{D.5.24}
$$

We then observe that (D.5.24) is actually the solution of the following state space (optimal filter).

$$
\frac{d\hat{x}(t)}{dt} = \left(\mathbf{A} - \mathbf{K_f}^T(t)\mathbf{C}\right)\hat{x}(t) + \mathbf{K_f}^T(t) y'(t) \tag{D.5.25}
$$

$$
\hat{x}(0) = \hat{x}_o \tag{D.5.26}
$$

$$
\hat{z}(t) = f^T x(t) \tag{D.5.27}
$$

We see that the final solution depends on $f$ only through (D.5.27). Thus, as predicted, (D.5.25), (D.5.26) can be used to generate an optimal estimate of any linear combination of states.

Of course, the optimal filter (D.5.25) is identical to that given in (22.10.23)

All of the properties of the optimal filter follow by analogy from the (dual) optimal linear regulator. In particular, we observe that (D.5.16) and (D.5.17) are a CTDRE and its boundary condition, respectively. The only difference is that, in the optimal-filter case, this equation has to be solved forward in time. Also, (D.5.16) has an associated CTARE, given by

$$\mathbf{Q} - \mathbf{\Sigma}_\infty \mathbf{C}^T \mathbf{R}^{-1} \mathbf{C} \mathbf{\Sigma}_\infty + \mathbf{\Sigma}_\infty \mathbf{A}^T + \mathbf{A} \mathbf{\Sigma}_\infty = \mathbf{0} \tag{D.5.28}$$

Thus, the existence, uniqueness, and properties of stabilizing solutions for (D.5.16) and (D.5.28) satisfy the same conditions as the corresponding Riccati equations for the optimal regulator.

Appendix E

# MATLAB SUPPORT

The accompanying disc contains a set of MATLAB-SIMULINK files. These files provide support for many problems posed in this book, and, at the same time, facilitate the study and application of selected topics.

| File name | Chapter | Brief description |
|---|---|---|
| **amenl.mdl** | *Chap. 19* | SIMULINK schematic to evaluate the performance of a linear design on a particular nonlinear plant. |
| **apinv.mdl** | *Chap. 2* | SIMULINK schematic to evaluate approximate inverses for a nonlinear plant. |
| **awu.mat** | *Chap. 26* | MATLAB data file – it contains the data required to use SIMULINK schematics in file **mmawu.mdl**. This file must be previously loaded to run the simulation. |
| **awup.m** | *Chap. 11* | MATLAB program to decompose a biproper controller in a form suitable to implement an anti-windup strategy – requires the function **p_elcero.m.** |
| **c2del.m** | *Chap. 3* | MATLAB function to transform a transfer function for a continuous-time system with zero-order hold into a discrete-transfer function in delta form. |
| **cint.mdl** | *Chap. 22* | SIMULINK schematic to evaluate the performance of a MIMO control loop in which the controller is based on state estimate feedback. |
| **css.m** | *Chap. 7* | MATLAB function to compute a one-d.o.f. controller for an $n^{th}$-order SISO, strictly proper plant (continuous or discrete) described in state space form. The user must supply the desired observer poles and the desired control poles. This program requires the function **p_elcero.m**. |
| **data_newss.m** | *Chap. 11* | MATLAB program to generate the data required for **newss.mdl** – this program requires **lambor.m**. |
| **dcc4.mdl** | *Chap. 10* | SIMULINK schematic to evaluate the performance of a cascade architecture in the control of a plant with time delay and generalised disturbance. |
| **dcpa.mdl** | *Chap. 13* | SIMULINK schematic to evaluate the performance of the digital control for a linear, continuous-time plant. |
| **dead1.mdl** | *Chap. 19* | SIMULINK schematic to study a compensation strategy for deadzones. |
| **del2z.m** | *Chap. 13* | MATLAB function to transform a discrete-time transfer function in delta form to its Z-transform equivalent. |

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| File name | Directory | Brief description |
|---|---|---|
| **dff3.mdl** | *Chap. 10* | SIMULINK schematic to evaluate the performance of disturbance feedforward in the control of a plant with time delay and generalised disturbance. |
| **distff.mdl** | *Chap. 10* | SIMULINK schematic to compare a one d.o.f. control against a two-d.o.f. control in the control of a plant with time delay. |
| **distffun.mdl** | *Chap. 10* | SIMULINK schematic to evaluate the performance of disturbance feedforward in the control of an unstable plant and generalised disturbance. |
| **lambor.m** | *Chap. 11* | MATLAB program to synthesise an observer – this routine can be easily modified to deal with different plants. |
| **lcodi.mdl** | *Chap. 13* | SIMULINK schematic to compare discrete-time and continuous-time PID controllers for the control of an unstable plant. |
| **linnl.mat** | *Chap. 19* | MATLAB data file, with the linear design data used in solved problem. |
| **mimo1.mdl** | *Chap. 21* | SIMULINK schematic with a motivating example for the control of MIMO systems. |
| **mimo2.mdl** | *Chap. 22* | SIMULINK schematic to simulate a MIMO design based on an observer plus state estimate feedback. |
| **mimo2.mat** | *Chap. 22* | MATLAB data file for **mimo2.mdl.** |
| **mimo3.mdl** | *Chap. 25* | SIMULINK schematic for the triangular control of a MIMO stable and nonminimum phase plant, by using an IMC architecture. |
| **mimo4.mdl** | *Chap. 26* | SIMULINK schematic for the decoupled control of a MIMO stable and minimum phase plant plant, using an IMC architecture. |
| **minv.m** | *Chap. 25* | MATLAB function to obtain the inverse (in state space form) of a **biproper** MIMO system in state space form. |
| **mmawe.mdl** | *Chap. 26* | SIMULINK schematic for the (dynamically decoupled) control of a MIMO system with input saturation – an anti-windup mechanism is used, and directionality is (partially) recovered by scaling the control error. |

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| File name | Directory | Brief description |
|---|---|---|
| **mmawu.mdl** | *Chap. 26* | SIMULINK schematic for the (dynamically decoupled) control of a MIMO system with input saturation – an anti-windup mechanism is used, and directionality is (partially) recovered by scaling the controller output. |
| **newss.mdl** | *Chap. 11* | SIMULINK schematic to study a (weighted) switching strategy to deal with state-saturation constraints. |
| **nmpq.mdl** | *Chap. 15* | SIMULINK schematic to evaluate disturbance compensation and robustness in the IMC control of a NMP plant. . |
| **oph2.m** | *Chap. 16* | MATLAB function to perform H2 minimization to solve the model-matching problem. |
| **p_elcero.m** | *Chap. 7* | MATLAB function to eliminate leading zeros in a polynomial. |
| **paq.m** | *Chap. 7* | MATLAB function to solve the pole assignment equation: The problem can be set either for Laplace transfer functions or by using the Delta-transform. This program requires the function **p_elcero.m.** |
| **phloop.mdl** | *Chap. 19* | SIMULINK schematic to evaluate the IMC control of a pH neutralisation plant by using approximate nonlinear inversion. |
| **phloop.mat** | *Chap. 19* | MATLAB data file associated **phloop.mdl** |
| **piawup.mdl** | *Chap. 11* | SIMULINK schematic to evaluate an anti-windup strategy in linear controllers, by freezing the integral action when its output saturates. |
| **pid1.mdl** | *Chap. 6* | SIMULINK schematic to analyze the performance of a PID control that uses empirical tuning methods. |
| **pidemp.mdl** | *Chap. 6* | SIMULINK schematic to use the Ziegler–Nichols tuning method based on closed-loop oscillation: The plant is linear, but of high order, with input saturation and noisy measurements. |
| **pmimo3.m** | *Chap. 25* | MATLAB program to compute the Q controller for solved problem. |
| **qaff1.mdl** | *Chap. 15* | SIMULINK schematic to analyze the loop performance of an IMC control loop of a NMP plant. |

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| File name | Directory | Brief description |
|---|---|---|
| **qaff22.mdl** | *Chap. 15* | SIMULINK schematic to analyze the loop performance of the Smith controller in Q form. |
| **qawup.mdl** | *Chap. 11* | SIMULINK schematic to implement an anti-windup mechanism in the IMC architecture – the decomposition of $Q(s)$ was done by using MATLAB function **awup.m.** |
| **sat_uns.mdl** | *Chap. 15* | SIMULINK schematic to study saturation in unstable plants with disturbances of variable duration. |
| **slew1.mdl** | *Chap. 11* | SIMULINK schematic to evaluate the performance of a PI controller with anti-windup mechanism to control a plant with slew-rate limitation. |
| **smax.m** | *Chap. 9* | MATLAB function to compute a lower bound for the peak of the nominal sensitivity $S_o$ – the plant model has a number of unstable poles, and the effect of one particular zero in the open RHP is examined. |
| **softloop1.mdl** | *Chap. 19* | SIMULINK schematic to compare the performances of linear and nonlinear controllers for a particular nonlinear plant. |
| **softpl1.mdl** | *Chap. 19* | SIMULINK schematic of a nonlinear plant. |
| **sugdd.mat** | *Chap. 24* | MATLAB data file: – it contains the controller required to do dynamically decoupled control of the sugar mill. |
| **sugmill.mdl** | *Chap. 24* | SIMULINK schematic for the multivariable control of a sugar mill station. |
| **sugpid.mdl** | *Chap. 24* | SIMULINK schematic for the PID control of a sugar mill station – the design for the multivariable plant is based on a SISO approach. |
| **sugtr.mat** | *Chap. 24* | MATLAB data file – it contains the controller required to do triangularly decoupled control of the sugar mill. |
| **tank1.mdl** | *Chap. 2* | SIMULINK schematic to illustrate the idea of inversion of a nonlinear plant. |
| **tmax.m** | *Chap. 9* | MATLAB function to compute a lower bound for the peak of the nominal complementary sensitivity $T_o$. The plant model has a number of NMP zeros, and the effect of one particular pole in the open RHP is examined. |

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|---|---|---|
| File name | Directory | Brief description |
| **z2del.m** | *Chap. 13* | MATLAB routine to transform a discrete-time transfer function in Z-transform form to its Delta-transform equivalent. |

**Table E.1.** Description of MATLAB support files